Some Spherical Diagrams over Labeled Oriented Trees and Graphs

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1 Introduction

A *labeled oriented graph* (LOG) is an oriented graph on vertices $\{1, ..., n\}$, where each oriented edge is labeled by a vertex. Associated with it comes a presentation on generators $x_1, \ldots, x_n$ in one-to-one correspondence with the vertices. For an edge with initial vertex i, terminal vertex j and label k we add a relation $x_i x_k = x_k x_j$. We refer to such a presentation as a LOG-presentation and to the standard 2-complex associated with it as a LOG-complex. We say a LOG is aspherical if its associated LOG-complex is aspherical. A LOT is a LOG, where the underlying graph is a tree. A LOI (labeled oriented interval) is a LOG where the underlying graph is an interval, i.e. each vertex has valence one or two. A LOF (labeled oriented forest) is a LOG where the underlying graph is a forest.

Let K be a finite 2-complex. In this article K will always be the standard 2-complex of a finite presentation. A *surface diagram* over K is a piecewise linear map $f\colon C \to K$, where C is a cell decomposition of a closed orientable surface and f carries open cells of C homeomorphically to open cells of K. If C is a 2-sphere then f is called a *spherical diagram*. A cell of C will be labeled by the cell of K it maps to under f. The 1-cells also get their orientation from the 1-cells of K. In this way C itself carries all the information of the map $f\colon C \to K$ and we often speak of the "diagram C". A surface diagram $f\colon C \to K$ is called *reducible* if there is a pair of 2-cells in C having a boundary edge t in common and being mapped by f onto the same 2-cell in K by folding over t. The surface diagram is called *reduced* if it is not reducible. A 2-complex K is called *diagrammatically reducible* (DR) if each spherical diagram over K is reducible.

Given a surface diagram $f\colon M \to K$ over a LOG-complex K we can draw its dual by replacing the square 2-cells by crossings. We undercross when labels change. The process is depicted for a single 2-cell of M in Figure 1. The orientation of the link is defined such that it matches with the orientation of

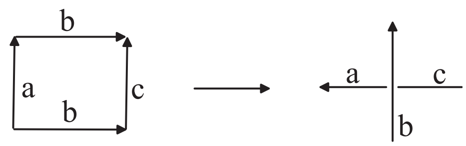

Figure 1: Dualizing a surface diagram

the crossings coming from the 2-cells in M. This leads to an oriented link projection L on the surface M which contains all the information of the diagram. It is an interesting property that L lifts to the LOG. In fact for any surface diagram over a LOG P there is a system of closed paths in P which come from the link. These paths are certainly contractible in P if P is a LOT. The example in the last section has contractible paths in a LOG which is not a LOT.

It is known that the question of whether a 2-complex is aspherical or not is recursively undecidable. Computer experiments for the special class of 2-complexes coming from LOTs have been carried out. A computer program has been running parallely on up to 300 computers for the past three years. It constructs reduced surface diagrams over LOTs. The program has checked roughly 60 billion LOTs, only checking those LOTs which are not known to be aspherical by other methods (see Rosebrock [5] for a survey on methods showing asphericity for LOTs). None of the LOTs can be proved to be aspherical by these computations, but the program found reduced spherical diagrams over about 100 different LOTs which have a certain, more general, structure than the diagrams in [4].

A labeled oriented graph is called *compressed* if every edge contains 3 different labels. It is called *boundary reducible* if there is a boundary vertex label that does not occur as edge label and *boundary reduced* otherwise. A labeled oriented graph is called *interior reducible* if there is a vertex with two adjacent edges with the same label that either point away or towards that vertex. A labeled oriented graph which is boundary reduced, interior reduced and compressed is called *reduced.* Any labeled oriented tree can be transformed into a reduced labeled oriented tree. The homotopy type of the associated 2-complex remains unchanged under this transformation.

In this paper we want to display and analyze how some of these diagrams are built. It is interesting to note that all examples of spherical diagrams found by the computer of complexity 3 or higher (see [6] for the definition of complexity) contain a reducible sub-LOT. That is the LOT contains a labeled sub-graph that is a LOT by itself (i.e. any edge-label of the sub-graph is also a vertex-label of the sub-graph).

2 Reduced Spherical Diagrams over LOTs

The first examples of LOTs that admit reduced spherical diagrams are given in [4]. They all have a special form. Each consists of parts $W_1, \ldots, W_n, V$ (n even) where each W_i is identified with V at exactly one vertex. All W_i are the same trees (isomorphic as graphs) with the same orientations and edge labels but different vertex labels. V is a sub-LOG of U, is a union of LOIs, and comes from a link projection. In most cases $n = 2$ and V is a LOI coming from a knot-projection. All these examples arise from the following construction (for details see [4]): Take a reduced surface diagram $f\colon M \to K$ over a LOT K where the strands going over the handles of M in the corresponding dual link L are all mapped to the same generator a. Cut along the handles and take each strand n times. If M has genus g this process leads to a map of the 2-sphere with $2g$ holes to a union of LOTs $W_1, \ldots, W_n$. Now glue in a link projection $2g$ times which maps to a union of LOIs V.

The fact that sub-LOTs play a key role is visible in the examples of reduced spherical diagrams found by computer search. All examples the computer has found are non-injective reduced LOIs. The examples found are either of complexity 2 (i.e. homotopy equivalent to a 1-relator presentation) or they contain a boundary-reducible sub-LOI. They all present knot groups. If the LOI has complexity 2, then the group is the infinite cyclic group, otherwise the group is the one presented by the reducible sub-LOI which is the Wirtinger-presentation of a knot.

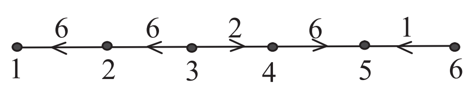

Figure 2: A non-DR LOI

Example 2.1 *The LOI of Figure 2 admits a reduced spherical diagram. This is the smallest known example over a reduced LOI. The LOI has 6 generators without a reducible sub-LOI and the spherical diagram consists of twelve 2-cells. It is of complexity 2 and therefore aspherical.*

The corresponding spherical diagram is depicted in Figure 3. It is reduced but will get reducible after a diamond move.

Example 2.2 *The LOI of Figure 4 admits a reduced spherical diagram.*

Note that this LOI contains a boundary-reducible sub-LOI V consisting of the vertices $\{5, 6, 7, 8\}$ and the edges between them. Taking out the sub-LOI, identifying vertices 8 and 5 and relabeling the vertices leads to Example 2.1[1]. This sub-LOI is a presentation of the trefoil and there are arguments implying:

[1]I thank the students of a seminar of Jens Harlander for this observation

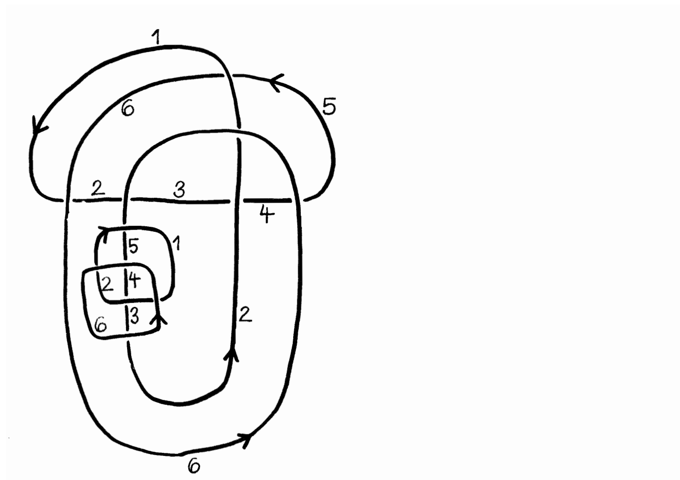

Figure 3: Spherical diagram

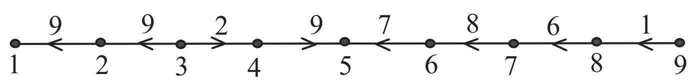

Figure 4: A second non-DR LOI

$x_1 = x_2 = x_3 = x_4 = x_5 = x_8 = x_9$. The LOI of Figure 4 records a Wirtinger presentation of the trefoil knot and has complexity 3. There is no pair of edges incident with the sub-LOI with the same label as in the examples of [4]. The corresponding spherical diagram is depicted in Figure 5.

Example 2.3 *The LOI of Figure 6 admits a reduced spherical diagram. It also contains a boundary-reducible sub-LOI.*

The corresponding spherical diagram is depicted in Figure 7. It is reduced but will get reducible after a diamond mopve along the dashed line.

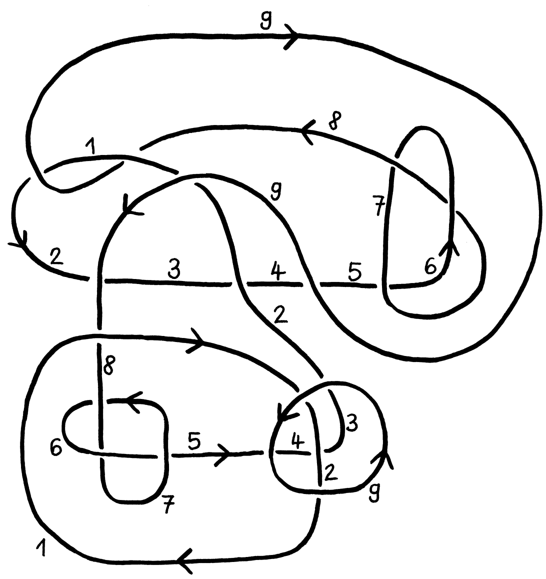

Figure 5: Spherical diagram

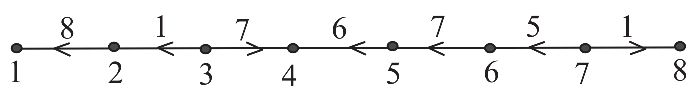

Figure 6: Another non-DR LOI

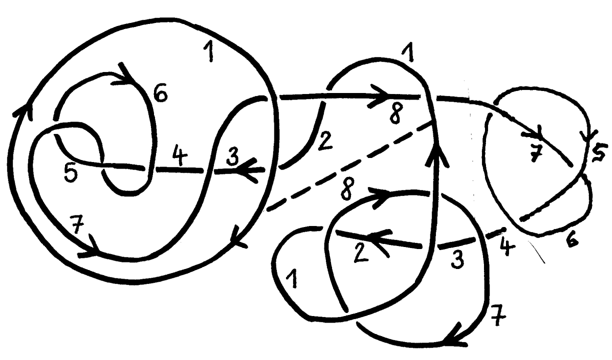

Figure 7: Spherical diagram

3 A LOG-example

We describe a reduced spherical diagram over a reduced LOG-presentation P with the following properties:

1. P has the Euler-Characteristic of a LOT,
2. all paths lifted into the LOG are contractible in the LOG,
3. the spherical diagram defines a non-trivial π_2-element.

Example 3.1 *The LOG of Figure 8 admits the reduced spherical diagram of Figure 9. The two paths in Figure 9 lift to contractible paths in the LOG. It corresponds to a non-trivial* π_2*-element because the two 2-cells of Figure 9 corresponding to the relator* $x_1x_4 = x_4x_5$ *are in different levels of the covering space corresponding to the map* $f\colon G \to \mathbb{Z}$ *which sends each* x_i *to 1.*

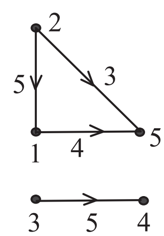

Figure 8: A LOG

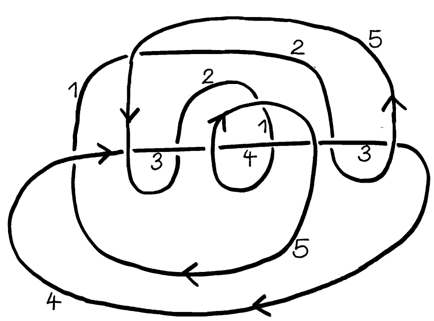

Figure 9: A corresponding spherical diagram

References

[1] J. Harlander, S. Rosebrock, *On Primeness of Labeled Oriented Trees*; Journal of Knot Theory and its Ramifications, Vol. 21 No. 8 (2012).

[2] J. Howie, *Some remarks on a problem of J.H.C. Whitehead*, Topology **22** (1983) 475-485.

[3] J. Howie, *On the asphericity of ribbon disc complements*, Trans A.M.S. **289**(1) (1985) 281-302.

[4] S. Rosebrock, *A reduced spherical diagram into a ribbon–disk complement and related examples*, Editor: P. Latiolais, Lecture notes in mathematics 1440, Springer Verlag (1990), 175-185.

[5] S. Rosebrock, *The Whitehead Conjecture - an overview*, Siberian Electronic Mathematical Reports 4 (2007), 440-449.

[6] S. Rosebrock, *On the Complexity of labeled oriented trees*, Proc. of the Indian Acad. of Sci 120 Nr. 1 (2010); S. 11 – 18.