relative

On the general Andrews-Curtis Conjecture

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Abstract

The general Andrews-Curtis Conjecture asks, whether two finite two-dimensional PLCW-complexes of the same simple-homotopy type can be three-deformed into each other. In this generality, unlike in higher dimensions, the conjecture was open up to now. is

In higher dimensions one can even add the requirements, that a fixed subcomplex between the first and the second complex can be kept fixed during the deformation, or that the final map is homotopic to the one given originally (C.T.C. Wall). These additional requirements don't hold in dimension 2, a fact which can be shown following A. Sieradski and almost unnoticedly passed attention. Technically it is based on two different ways of calculating the **bias** of appropriate 2-complexes, once via diagonal maps of fundamental groups, once via Q-transformations.

By suitably enlarging the Sieradski -examples, we hope to get rid of the unwanted requirements, thus ending with a negative answer for the general Andrews-Curtis Conjecture without additional restrictions.

(A note on History: Since several decades I have expected a negative outcome of the Conjecture. But two years ago I committed the sin of turning my mind. Friends and colleagues turned it back again. I am very grateful to them.)

1. Bias

The first examples of 2-complexes having the same finite abelian fundamental group and Euler characteristic but different (simple-) homotopy types could be distinguished by the so-called bias invariant. It concerned how spherical elements lie in the second homology of the complexes. 1) Alternatively one can count exponents of generators in commutators amongst defining relators. 2)

In his paper [Si77] Allan Sieradski at once treated free products of finite abelian groups by use of the same technique. A simple example with the bias modulus m=5 is contained in a Chapter on the Andrews-Curtis Conjecture and its Generalizations written by Cynthia Hog-Angeloni and myself in [LMS 197, p.377-378]. We copy it from there: 3)

from p. 377

Let $\mathcal{P}_1$ resp. $\mathcal{Q}_1$ be presentations of $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ given by $\langle a_1, a_2, a_3 | a_1^5, a_2^5, a_3^5, [a_1, a_2], [a_1, a_3], [a_2, a_3] \rangle$ resp. $\langle a_1, a_2, a_3 | a_1^5, a_2^5, a_3^5, [a_1^2, a_2], [a_1, a_3], [a_2, a_3] \rangle$. $\mathcal{P}_2$ resp. $\mathcal{Q}_2$ are defined as $\mathcal{P}_1$ resp. $\mathcal{Q}_1$, but with generators b_i instead of the a_i.

from p. 378

(17) $\mathcal{P}_1 + \mathcal{P}_2$ can be Q-transformed into $\mathcal{Q}_1 + \mathcal{Q}_2$

by the following chain of semisplit Q-transformations:

$$\begin{array}{cccccc}
a_1^5 = 1 \;, & a_2^5 = 1 \;, & [a_1, a_2] = 1 \;, & b_1^5 = 1 \;, & b_2^5 = 1 \;, & [b_1, b_2] = 1 \\
\multicolumn{6}{c}{\downarrow} \\
a_1^5 = a_2^5 \;, & a_2^5 = [a_1^2, a_2] \;, & '' \;, & b_1^5 = b_2^5 \;, & b_2^5 = [a_1^2, a_2] \;, & '' \\
\multicolumn{6}{c}{\downarrow (\alpha)} \\
'' \;, & '' \;, & [a_1^2, a_2]^3 = 1 \;, & '' \;, & '' \;, & '' \\
\multicolumn{6}{c}{\downarrow} \\
'' \;, & '' \;, & \mathbf{b_2^{15} = 1} \;, & '' \;, & '' \;, & '' \\
\multicolumn{6}{c}{\downarrow} \\
'' \;, & '' \;, & b_2^{15} = [b_1^2, b_2] \;, & '' \;, & '' \;, & '' \\
\multicolumn{6}{c}{\downarrow (\beta)} \\
'' \;, & '' \;, & '' \;, & '' \;, & '' \;, & [b_1^2, b_2]^3 = 1 \\
\multicolumn{6}{c}{\downarrow} \\
'' \;, & '' \;, & '' \;, & '' \;, & '' \;, & b_2^{45} = 1 \\
\multicolumn{6}{c}{\downarrow} \\
'' \;, & '' \;, & '' \;, & '' \;, & '' \;, & b_2^5 = [a_1^2, a_2]^{15} \\
\multicolumn{6}{c}{\downarrow (\gamma)} \\
'' \;, & '' \;, & '' \;, & '' \;, & '' \;, & b_2^5 = 1 \\
\multicolumn{6}{c}{\downarrow} \\
'' \;, & '' \;, & [b_1^2, b_2] = 1 \;, & b_1^5 = 1 \;, & [a_1^2, a_2] = 1 \;, & '' \\
\multicolumn{6}{c}{\downarrow}
\end{array}$$