A generalized metric small cancellation condition and Artin groups

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Small cancellation

Definition

Given $P = \langle X \mid R \rangle$ we consider $S(R)$ the closure of R with respect to taking cyclic conjugates and inverses. A reduced word $u \in F(X)$ is a *piece* if there exist distinct $r_1, r_2 \in S(R)$ that have u as a common prefix.

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Example

Let $P = \langle a, b, c, x, y, z \mid abcxyz, zyxabc \rangle$. Then abc is a piece, since

$$abcxyz, abczyx \in S(R).$$

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Sea $P = \langle X \mid R \rangle$

- Let $0 < \lambda < 1$. A presentation P satisfies condition $C^{\prime}(\lambda)$ if whenever a piece u is a subword of $r \in S(R)$, then $|u| < \lambda |r|$.

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- Let $0 < \lambda < 1$. A presentation P satisfies condition $C'(\lambda)$ if whenever a piece u is a subword of $r \in S(R)$, then $|u| < \lambda |r|$.
- Let $p \in \mathtt{N}$. A presentation P satisfies condition $C(p)$ if no element of $S(R)$ can be expressed as the product of less than p pieces.

Diagrams

Definition

Let $P = \langle X \mid R \rangle$. A *diagram* Δ over P is a combinatorial function $\varphi : M \to K_P$, where M is a combinatorial structure of the sphere (possibly without some open 2-cells), and K_P is the presentation complex.

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Definition

A diagram Δ is called *reducible* if it has two 2-cells $f, f^{\prime}$ such that the intersection of their boundaries has an edge e such that the words read on their boundaries with opposite orientations starting at a vertex of e coincide. Otherwise, Δ is called *reduced*.

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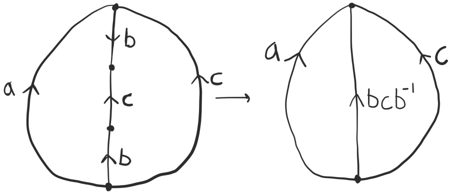

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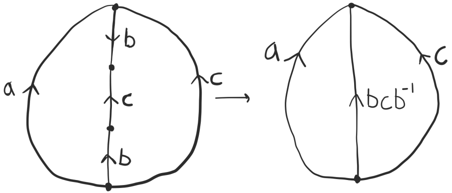

Note that the words that appear in the interior edges of a reduced diagram over a presentation P are pieces of P.

Diagrams

Definition

Let $P = \langle X|R\rangle$ be a finite presentation of a group G. Given $w \in F(X)$ cyclically reduced, w trivial in G, we define $\mathrm{Area}(w)$ as the least amount of 2-cells in a reduced diagram with boundary w. We define the *Dehn function* of P as

$$\mathrm{Dehn}(n) = \max\{\mathrm{Area}(w) : w = 1 \text{ en } G, |w| \leq n, w \text{ reduced}\}.$$

Classic results

Theorem

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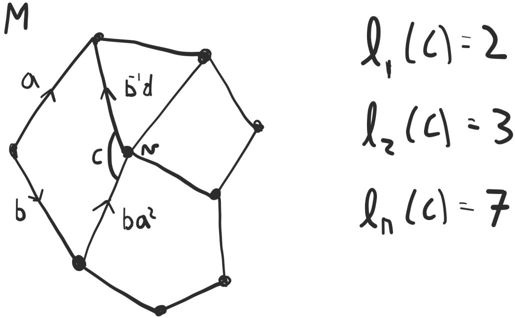
Theorem

Let P be a presentation without proper powers of a group G. If P is $C(6)$, $C(4)$-$T(4)$ or $C(3)$-$T(6)$, then K_P is aspherical.

Condition $\mathcal{T}'$

Let P be a presentation and M a diagram over P. For each interior vertex v and every edge c in the link of v we define $l_1(c)$ and $l_2(c)$ as the lengths of the words read in the two edges that form c, and $l_r(c)$ as the length of the word read on the corresponding face.



Condition $\mathcal{T}'$

With the previous notation we define

$$d^0_F(v) = \sum_{c \ni v} \frac{l_1(c) + l_2(c)}{l_r(c)},$$

where the sum is over all edges in the link of v.

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$$d'_F(v) = \sum_{c \ni v} \frac{l_1(c) + l_2(c)}{l_r(c)},$$

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Definition

We say a presentation P satisfies condition $\mathcal{T}'$ if for every interior vertex of any reduced diagram over P, $d'_F(v) \leq d(v) - 2$. If this inequality is strict, we sat P satisfies condition $\mathcal{T}'_<$.

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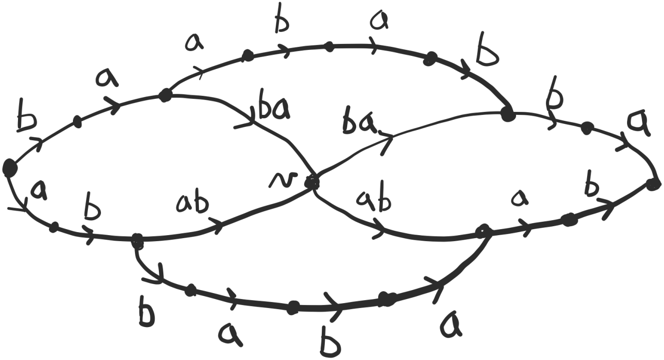
Note that condition $\mathcal{T}'_<$ generalizes conditions $C'(\frac{1}{6})$ and $C'(\frac{1}{4})$-$T(4)$.

Condition $\mathcal{T}'$

$$P = \langle a, b \mid ababa^{\ 1}b^{\ 1}a^{\ 1}b^{\ 1}\rangle$$



$$d_F^0(v) = \frac{2+2}{8} + \frac{2+2}{8} + \frac{2+2}{8} + \frac{2+2}{8} = 2 = d(v) - 2$$

Examples

Example

The following cyclic presentation satisfies condition $\mathcal{T}_{<}^{0}$ but is not $C(6)$ nor $T(4)$.

$$\langle x_0, x_1, x_2, x_3, x_4 \mid x_{i+4}^{1} x_{i+1}^{1} x_i^{\ 1} (x_{i+4} x_{i+1})^2 \text{ for } i = 0, \ldots, 4 \rangle$$

Example

The following cyclic presentation satisfies condition $\mathcal{T}^{0}$.

$$\langle x_0, \ldots, x_6 \mid x_{i+1} x_i^{\ 1} x_{i+6} x_{i+1}^{1} x_i x_{i+6}^{1} x_{i+2}^{1} \text{ for } i = 0, \ldots, 6 \rangle$$

However the presentation is not $\mathcal{T}_{<}^{0}$, nor $C(6)$, $C(4)$-$T(4)$ or $C(3)$-$T(6)$.

Condition $\mathcal{T}'$

Theorem [B-Minian-Sadofschi Costa]

Let P be a presentation that satisfies condition $\mathcal{T}'$ and has no proper powers. Then K_P is aspherical. Moreover, P is diagrammatically reducible.

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Sketch of proof

Let $\varphi\colon M \to K_P$ be a reduced spherical diagram.

$$2 = \mathcal{V}(M) - \mathcal{E}(M) + \mathcal{F}(M) =$$

$$\mathcal{V}(M) - \frac{1}{2} \sum_{v 2 \mathrm{vertices}(M)} d(v) + \frac{1}{2} \sum_{v 2 \mathrm{vertices}(M)} d^{0}_{F}(v) \leq 0,$$

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Theorem [B-M-SC]

Let G be a group that admits a finite presentation which satisfies conditions $\mathcal{T}^{\emptyset}_{<}$-$C(3)$. Then G is hyperbolic.

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Theorem [B-M-SC]

Let P be a presentation which satisfies conditions $\mathcal{T}'$-$C'(\frac{1}{2})$ such that all its relators have the same length. Then P has quadratic Dehn function. Moreover, if P is finite the word and conjugacy problems are solvable.

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Theorem [B-M-SC]

Let G be a group that admits a finite presentation which satisfies conditions $\mathcal{T}^{\emptyset}_{<}$-$C(3)$. Then G is hyperbolic.

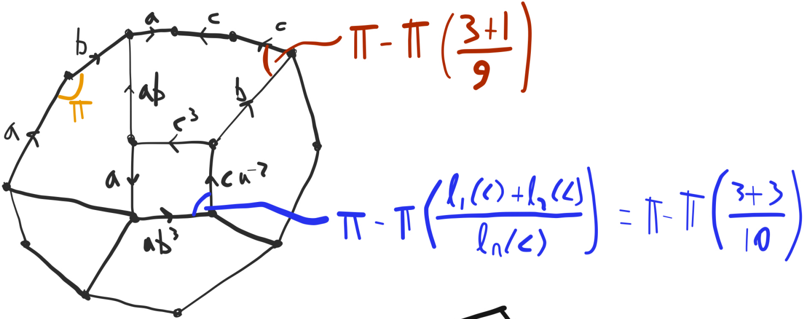
Theorem [B-M-SC]

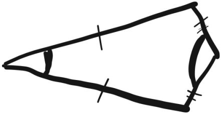
Let P be a presentation which satisfies conditions $\mathcal{T}^{\emptyset}$-$C^{\emptyset}(\frac{1}{2})$ such that all its relators have the same length. Then P has quadratic Dehn function. Moreover, if P is finite the word and conjugacy problems are solvable.

Theorem [B-M-SC]

Let $P = \langle X \mid R \rangle$ be a presentation of a group G, satisfying conditions $\mathcal{T}^{\emptyset}$-$C^{\emptyset}(\frac{1}{2})$. Let $r_{\min}$ be the length of the shortest relator. Then any nontrivial word W in the free group generated by X representing the trivial element in G has length at least $r_{\min}$.

Angles





Curvature

$$\kappa(v) = 2\pi - \pi\chi(\mathrm{lk}_v) - \sum_{c \ni v} w(c),$$

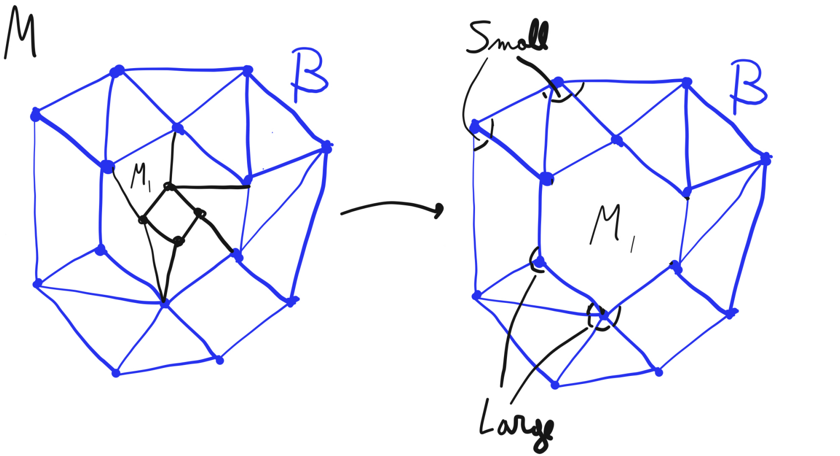
$$\kappa(f) = 2\pi - \pi\ell(\partial f) + \sum_{c \in f} w(c),$$

Theorem (Combinatorial Gauss–Bonnet Theorem)

Let K be an angled 2-complex. Then

$$\sum_{f \in \mathrm{faces}(K)} \kappa(f) + \sum_{v \in \mathrm{vertices}(K)} \kappa(v) = 2\pi\chi(K).$$

Sketch of proof



$$V(\partial M_1) < V(\partial M)$$

Artin groups

Definition

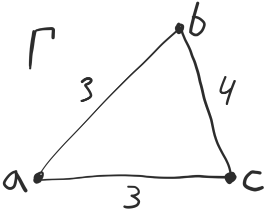
Let Γ be a finite simple graph with edges labeled by integers greater than 1. The *Artin group defined by* Γ is the group A_Γ given by the following presentation P_Γ. Its generators correspond to the vertices of Γ, and there is a relator of the form

$$\underbrace{ababa\cdots}_{m \text{ letters}} = \underbrace{babab\cdots}_{m \text{ letters}}$$

for each pair of vertices a and b connected by and edge labeled by m.

Artin groups

Example



$$P_\Gamma = \langle a, b, c \mid aba(bab)^{-1}, aca(cac)^{-1}, cbcb(bcbc)^{-1} \rangle$$

Artin groups

Theorem [Charney-Davis]

An Artin group A_Γ has cohomological dimension at most 2 if and only if for every triangle in the graph Γ with edges labeled by p, q and r we have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$.

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Corollary

The presentation complex of a presentation P_Γ of an Artin group is aspherical if and only if P_Γ is diagrammatically reducible.