Combinatorial Relative Asphericity

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Questions

Joint work with J. Harlander

Let L be a 2-complex and $K \subset L$.

We ask:

1. Under which assumptions does $\pi_1(K)$ embed into $\pi_1(L)$?
2. If K is aspherical under which assumptions is L aspherical?

Questions are related.
1. leads to Freieheitssatz-type of theorems.
2. was interesting also because of:

Theorem: Huck/Rb (2001): If a compressed injective LOT P does not contain a boundary-reducible Sub-LOT then the LOT-complex $K(P)$ is DR.

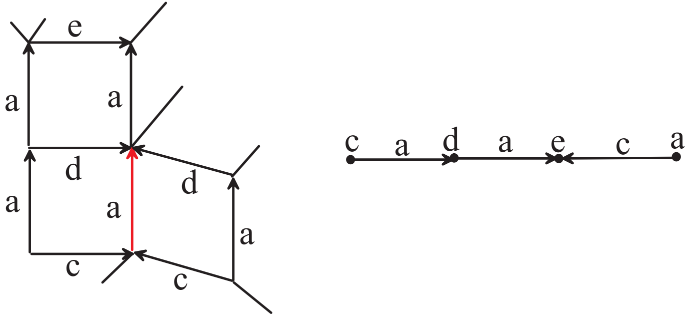
Spherical diagrams

Let K be a 2-complex. $f\colon C \to K$ is a spherical diagram, if C is a cell decomposition of the 2-sphere and open cells are mapped homeomorphically.

If K is non-aspherical then there exists a spherical diagram which realizes a nontrivial element of $\pi_2(K)$.

Reducibility

A spherical diagram $f\colon C \to K$ is called reducible, if there is a pair of 2-cells in C, which are mapped to K by folding over a common edge, the folding edge.



DR

If a spherical diagram $f\colon C \to K$ is reducible, then there is a spherical diagram of the same homotopy type without this pair of 2-cells.

If each spherical diagram over K is reducible, then K is called diagrammatically reducible (DR).

If K is DR then K is aspherical. (Any spherical diagram can be reduced until it is empty.)

Directed Diagrammatic Reducibility:

Definition: (HARLANDER/RB) Let K be a 2-complex with edge set X. Let Y be a proper subset of X (i.e. $X \neq Y$). We say that K is

- *DR directed away from* Y if every spherical diagram $f\colon C \to K$ that contains an edge with label from $X - Y$ also contains a folding edge with label from $X - Y$;
- *DR in all directions* if every spherical diagram $f\colon C \to K$ that contains an edge labeled $x \in X$ also contains a folding edge with label x. Note that this implies that K is DR directed away from all proper $Y \subset X$.

Directed Diagrammatic Reducibility

Example: Orientable closed surfaces are DR in all directions.

$$P = \langle x_1, x_2, \ldots, x_{2g-1}, x_{2g} \mid [x_1, x_2] \ldots [x_{2g-1}, x_{2g}] \rangle$$

and let $f\colon C \to K(P)$ be a spherical diagram that contains an edge x_1. If d is a 2-cell in C with two of its edge labels x_1, then draw a red line in d connecting the midpoints of the x_1-edges.

Red graph is a union of circles in C. Take one circle, the 2-cells form an annulus. Both boundary words u, v are from $\{x_2, \ldots, x_{2g}\}$.

Subgroup generated by $\{x_2, \ldots, x_{2g}\}$ is free, so u, v contain a cancelling pair. The corresponding folding edge is labelled x_1. (Same argument with other generators)

Directed Diagrammatic Reducibility

Let K be a 2-complex with edge set X and $Y \subset X$ proper. Let $K_Y \subset K$ have edge set Y and 2-cells of K with all boundary 1-cells from Y.

Theorem: Suppose that K is DR directed away from $Y \subset X$. Then

1. $\pi_2(K)$ is generated (as a $\pi_1(K)$-module) by the image of the inclusion induced map $\pi_2(K_Y) \to \pi_2(K)$,
2. every disc diagram $g\colon D \to K$ with boundary labeled by a word in Y, that contains a label from $X - Y$, has a folding edge with label from $X - Y$. Consequently, the inclusion induced map $\pi_1(K_Y) \to \pi_1(K)$ is injective.

Proof of 1. $\pi_2(K)$ is generated (as a $\pi_1(K)$-module) by the image of the inclusion induced map $\pi_2(K_Y) \to \pi_2(K)$:

Suppose $f: C \to K$ is a reduced spherical diagram. If $f(C)$ is not contained in K_Y then C contains an edge e so that $f(e) \notin Y$.

K is DR directed away from Y implies C contains a folding edge e so that $f(e) \notin Y$, contradicting that $f: C \to K$ is reduced. Since $\pi_2(K)$ is generated (as a $\pi_1(K)$-module) by reduced spherical diagrams, the first statement follows. □

Proof of 2. Every disc diagram $g\colon D \to K$ with boundary labeled by a word in Y, that contains a label from $X - Y$, has a folding edge with label from $X - Y$:

Suppose $g\colon D \to K$ is a disc diagram as in statement (2). We double D and construct a spherical diagram $g'\colon C = D_1 \cup D_2 \to K$, where D_1 is mapped by g and D_2 is mapped by $-g$.

C contains an edge with label not in Y. Since K is DR away from Y this spherical diagram contains a folding edge with label not in Y. This folding edge can not occur on $\partial D_1 = \partial D_2$. Thus D_1 or D_2 contain an interior folding edge with label not in Y and hence so does D. □

Freiheitssatz

Classical Freiheitssatz: $P = \langle x_1, \ldots, x_n \mid r \rangle$, r is a cyclically reduced word that contains all the generators. Then any proper subset S of $\{x_1, \ldots, x_n\}$ generates a free subgroup of $G(P)$ with basis S.

Freiheitssatz

Let $P = \langle X \mid R \rangle$. $Y \subset X$ define P_Y to be the subpresentation of P with generators Y and all relators that contain only generators of Y.

Theorem: Let $P = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ be DR in all directions. Then the inclusion induced homomorphism $G(P_Y) \to G(P)$ is injective for every subset Y of the generators.

Proof. If Y is the set of generators of P then $G(P_Y) = G(P)$ and the statement is true. If Y is a proper subset of the set of generators we use the Theorem stated before. $\square$

Freiheitssatz

Theorem: Let $P = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ and Y a proper subset of the generators. Assume that each r_i contains a generator not from Y. If P is DR directed away from Y, then Y generates a free subgroup of $G(P)$ with basis Y.

Proof. Since each r_i contains a generator not from Y we have that $P_Y = \langle Y \mid \rangle$ and $G(P_Y)$ is free. The above theorem (2) gives the desired result. □

Freiheitssatz

Here is a strengthening of the classical Freiheitssatz:

Theorem: Let $P = \langle x_1, \ldots, x_n \mid r \rangle$ be a one-relator presentation where r is a cyclically reduced relator that is not a proper power. Then P is DR in all directions.

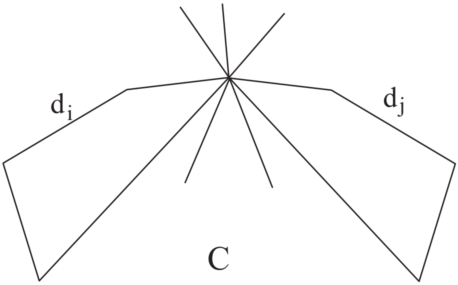
The proof uses a left ordering of the group which exists by theorems of Burns-Hale and Brodskii.

There is a notion of diagrammatic reducibility for relative presentations by B. Bogley and S. Pride, which is equivalent to directed diagrammatic reducibility.

Relative Vertex Asphericity

There is another relative notion:

(HUCK, RB) A spherical diagram $f\colon C \to K$ is called vertex reducible, if there is a pair of 2-cells $d_i, d_j \in C$, which are mapped to the same 2-cell of K by folding over a common vertex, the folding vertex. In that case the pair of 2-cells (d_i, d_j) is called a folding pair.



Relative Vertex Asphericity

f is called vertex reducible if it has a folding vertex. A 2-complex L is called vertex aspherical (VA) if each spherical diagram over L has a folding vertex.

K is DR $\Rightarrow$ K is VA $\Rightarrow$ K is aspherical.

Relative Vertex Asphericity

Definition (HARLANDER/RB) Let L be a 2-complex and $K \subset L$. L is VA relative to K if every spherical diagram $f\colon C \to L$, $f(C) \not\subseteq K$, has a folding vertex with folding pair of 2-cells in $L - K$.

Definition (HARLANDER/RB) Let L be a 2-complex and $K \subset L$. L is DR relative to K if every spherical diagram $f\colon C \to L$, $f(C) \not\subseteq K$, has a folding edge with folding pair of 2-cells in $L - K$.

Relative Vertex Asphericity

L is DR relative $K \Rightarrow L$ is VA relative K.

Theorem: Let L be a 2-complex and K a subcomplex. If K is VA (DR) and L is VA (DR) relative to K then L is VA (DR).

Proof (of the VA case). Assume $f\colon C \to L$ is a vertex reduced spherical diagram. Since L is VA relative to K we have that $f(C) \subseteq K$. So $f\colon C \to K$ is a vertex reduced spherical diagram, contradicting the assumption that K is VA. $\square$

Relative Vertex Asphericity

Theorem: If L is VA relative to K, then $\pi_2(L)$ is generated, as $\pi_1(L)$-module, by the image of $\pi_2(K)$ under the map induced by inclusion. In particular, if K is aspherical, then so is L.

Proof. Every vertex reduced spherical diagram $f\colon C \to L$ has its image $f(C)$ in K. Thus f represents an element in $\pi_2(K)$. Since $\pi_2(L)$ is generated by vertex reduced spherical diagrams, it follows that $\pi_2(L)$ is generated by the image of $\pi_2(K)$. □

DDR implies Relative VA

Theorem: Let K be a 2-complex with edge set X and $Y \subset X$. If K is DR directed away from Y, then K is DR relative to K_Y.

Proof. Assume K is not DR relative to K_Y. Then there exists a spherical diagram $f\colon C \to K$ such that $f(C) \not\subseteq K_Y$ where all pairs of 2-cells which may be reduced lie in K_Y.
Since $f(C) \not\subseteq K_Y$ we have that C contains an edge labelled by an element of $X - Y$. Since all pairs of 2-cells which may be reduced lie in K_Y we have that all folding edges of C are labelled by elements of Y. So K is not DR directed away from Y. □

Relative VA does not imply DDR

Let P=$\langle a, b, c \mid bac^{-1}, cb^{-1}a^{-1}\rangle$ be a presentation. $K(P)$ is the torus. There is a disk diagram D with boundary reading $aba^{-1}b^{-1}$ achieved by gluing the two relator disks along c together. Glue D to $-D$ to obtain a spherical diagram which is reducible at a and b only. This shows that P is not DR away from $Y = \{a, b\}$.

Since K_Y is the 2-complex modeled on the presentation $\langle a, b \mid \rangle$ which has no relators we have that $K(P)$ is DR relative K_Y if and only if $K(P)$ is DR. But this presentation of the torus is certainly DR, so $K(P)$ is DR relative K_Y.

Let K be a standard 2-complex. The link $\mathrm{lk}(K)$, is the boundary of a regular neighborhood of the vertex in K. (Whiteheadgraph)

The vertices of $\mathrm{lk}(K)$ are $\{x^+, x^- \mid x \in X\}$, where x^+ is a point of the oriented edge x close to the beginning, and x^- is a point close to the ending of that edge.

The positive link $\mathrm{lk}^+(K)$ is the *full subgraph* on the vertex set $\{x^+ \mid x \in X\}$ and the negative link $\mathrm{lk}^-(K)$ is the full subgraph on the vertex set $\{x^- \mid x \in X\}$.

Forest implies aspherical

Let C be a cell decomposition of the 2-sphere with oriented 1-cells. A source in C is a vertex with all its adjacent edges point away from it. A sink is a vertex with all its adjacent edges point towards it.

A 2-cell $d \in C$ is said to have exponent sum 0 if, when traveling along the boundary of d in clockwise direction, one encounters the same number of positive as of negative edges.

Theorem: (Gersten) Let C be a cell decomposition of the 2-sphere with oriented edges, such that all 2-cells have exponent sum 0. Then C contains a sink and a source.

Forest implies aspherical

Theorem: (Gersten) Let L be a finite 2-complex with one vertex and attaching maps of 2-cells in L have exponent sum 0. If $\mathrm{lk}^+(L)$ is a forest or $\mathrm{lk}^-(L)$ is a forest then L is DR.

Proof. Let $f\colon C \to L$ be a spherical diagram. 2-cells of C have exponent sum 0. Last Theorem implies in C is a sink and a source. A source leads to a cycle in $\mathrm{lk}^+(L)$. If $\mathrm{lk}^+(L)$ is a forest the cycle is reducible and hence the diagram is reducible. Likewise with a sink and a cycle in $\mathrm{lk}^-(L)$. □

Relative forest implies relative aspherical

A relative version:

Let Γ be a graph and $\hat{\Gamma} = \Gamma_1 \cup \ldots \cup \Gamma_n$ be a union of disjoint subgraphs. We write $\Gamma/\hat{\Gamma}$ for the graph obtained from Γ by collapsing each Γ_i to a vertex.

Γ is a forest relative to $\hat{\Gamma}$ if $\Gamma/\hat{\Gamma}$ has no cycles.

Theorem: Let L be a finite 2-complex with one vertex, $K \subseteq L$ and attaching maps of 2-cells in L have exponent sum 0. If $\mathrm{lk}^+(L)$ is a forest relative to $lk^+(K)$ or $\mathrm{lk}^-(L)$ is a forest relative to $\mathrm{lk}^-(K)$ then L is VA relative to K. Furthermore, the inclusion induced homomorphism $\pi_1(K) \to \pi_1(L)$ is injective.

Relative forest implies relative aspherical

Proof (of the first claim). Assume $\mathrm{lk}^+(L)$ is a forest relative to $\mathrm{lk}^+(K)$. Assume there is a vertex reduced spherical diagram $f\colon C \to L$, $f(C) \not\subseteq K$. Replace maximal regions in C which map to K by disks. C contains a source and a sink. Since $\mathrm{lk}^+(L)/\mathrm{lk}^+(K)$ is a tree, a vertex reduction is possible at a source $v \in C$. □

Weight test

Weight test (Gersten):

Let $P = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ be a presentation.
Assume there is a function

$$w\colon \{\text{Edges of } W(P)\} \to \mathbb{R}$$

such that

1. $\sum_{e \in z} w(e) \geq 2$ for all reduced cycles $z \in W(P)$.
2. $\sum_{e \in r_i} w(e) \leq d_i - 2$ for all relators r_i (d_i is length(r_i)).

Then $K(P)$ is DR.

Weight test

Proof: Assume $f\colon C \to K(P)$ is a reduced spherical diagram.

Pull back the weights of $W(P)$ to C.

C reduced implies $\sum_{e \text{ at } v} w(e) \geq 2$. Which implies $\sum_{e \in C} w(e) \geq 2V$ (V: number of vertices in C).

Second condition ($\sum_{e \in r_i} w(e) \leq d_i - 2$) implies:
$\sum_{e \in C} w(e) \leq -2F + \sum d_i = -2F + 2E$
(F number of 2-cells, E number of edges in C).

Combine: $2V \leq -2F + 2E$
Equivalent to $2F - 2E + 2V \leq 0$
Contradiction to the Euler-Characteristic of the 2-sphere. □

Relative Weight tests

Let Γ be a graph and $z = e_1 ... e_q$ a cycle in Γ. We say z is homology reducible if it contains a pair or edges e_i, e_j such that $e_i = \bar{e}_j$ (the bar indicates opposite orientation) and homology reduced otherwise.

If a given spherical diagram $f\colon C \to K$ is vertex reducible then there is link of a vertex in C with homology reducible image.

Let L be a 2-complex and $K = K_1 \vee \ldots \vee K_n \subset L$. Let $W(L, K)$ be $\mathrm{lk}(L)$ where $\mathrm{lk}(K_i)$ is replaced by two vertices k_i^+, k_i^- and exactly one edge e_i connecting these two vertices for each $1 \leq i \leq n$.

Theorem: Let L be a 2-complex with one vertex and cyclically reduced attaching maps of 2-cells and let $K = K_1 \vee \ldots \vee K_n \subset L$ such that no edge of K represents the trivial element in $\pi_1(K)$. Let ω be a weight function on the edges of $W(L, K)$ such that:

1. $\sum_i \omega(c_i) \leq q - 2$ if $c_1, \ldots, c_q$ are the corners of a 2-cell of $L - K$,
2. if z is a homology reduced cycle in $W(L, K)$ then $\omega(z) \geq 2$,
3. $\omega(e_i) = 0$ for $1 \leq i \leq n$,

then L is VA relative K.

Relative Weight tests

Proof: Assume $f\colon C \to L$ is a vertex reduced spherical diagram such that $f(C) \not\subset K$. Pull back the weights of $W(L, K)$ to corners in 2-cells of C which map to $L - K$.

Replace each maximal region $d \in C$ which is mapped to a single K_i by a 2-cell d' (we call d' a *replaced 2-cell*) and achieve a new cell decomposition C' of the 2-sphere.

Assign weight 0 to all corners of d'. Assign the weights of C to the corresponding corners of non-replaced 2-cells of C'.

The curvature of those 2-cells of C' coming from 2-cells of C mapped to $L - K$ have curvature less or equal to 0 by condition 1.

A replaced 2-cell $d' \in C'$ has at least two corners by the condition that no edge of K represents the trivial element in $\pi_1(K)$. It has weight 0 and condition 1 is satisfied for this 2-cell, leading also to non-positive curvature for replaced 2-cells.

Relative Weight tests

Assume z' is the link of a vertex in C'. If z' contains no corners of replaced 2-cells then it has weight at least two by condition 2 since f is homology reduced. If z' contains a corner of a replaced 2-cell, this corner will contribute 0 to the weight of z'. Since it does not appear in $W(L, K)$ if it is a corner in $\mathrm{lk}^+(K_i)$ or $\mathrm{lk}^-(K_i)$ or it has weight 0 in $W(L, K)$ we have weight at least two by condition 2 for z'.

So we have non-positive curvature at vertices of C' contradicting the Euler-Characteristic of C' by the combinatorial Gauss-Bonet theorem. □

Relative Weight tests

Here is a weight test for DDR:

Theorem: Let K be a 2-complex with cyclically reduced attaching maps of 2-cells and edge set $X \cup Y$ with $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_p\}$. Suppose we can assign weights $\omega(e) \geq 0$ to the edges e of $\text{lk}(K)$, such that:

1. If e connects y_i^ϵ with y_j^δ, $(\epsilon, \delta = \pm)$ then $\omega(e) \geq 1$;
2. If one of e's boundary vertices is y_i^+ or y_i^-, then $\omega(e) \geq 1/2$;
3. If z is a reduced cycle in $\text{lk}(K)$, then $\omega(z) \geq 2$;
4. Let d be a 2-cell of length $\kappa(d)$ from K, then $\sum_{c \in d} \omega(c) \leq \kappa(d) - 2$.

Then K is DR directed away from Y.

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