LOT's of Coxeter Type

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Wirtinger (LOT) presentations

Deficiency 1 Wirtinger presentations define knot groups, long virtual knot groups, and ribbon 2-knot groups. They are encoded via *labeled oriented trees*. A LOT is an oriented finite tree Γ on vertices $\mathbf{x}$ and edges $\mathbf{e}$, where each oriented edge is labeled by a word in $\mathbf{x}^{\pm 1}$. Associated with a LOT Γ is the presentation

$$P(\Gamma) = \langle \mathbf{x} \mid \mathbf{r} = \{r_e \mid e \in \mathbf{e}\}\rangle,$$

where $r_e : xw = wy$ in case $e = (x \xrightarrow{w} y)$ is the edge of Γ starting at x, ending at y, and labeled with the word w in $\mathbf{x}^{\pm 1}$. We denote by $K(\Gamma)$ and $G(\Gamma)$ the standard 2-complex and the group defined by $P(\Gamma)$, respectively.

The complex $K(\Gamma)$ is a subcomplex of a contractible complex.

Whitehead's asphericity conjecture for LOT's: $K(\Gamma)$ is aspherical.

If $G(\Gamma)$ is a 1-relator group then $K(\Gamma)$ is aspherical. Many prime knot groups are 1-relator groups, but not all.

1. How do we know $rk(G(\Gamma)) \geq 3$, and hence $G(\Gamma)$ is not 1-relator? Use Coxeter quotients!
2. Are LOT's of Coxeter type aspherical?

Coxeter presentations

Let Υ be a simplicial graph on vertices $\mathbf{x}$, and suppose edges e are labeled with integers $m_e \geq 1$. Define

$$P(\Upsilon) = \langle \mathbf{x} \mid x^2, \in \mathbf{x}, (xy)^{m_e} \text{ if } e = \{x, y\} \text{ is an edge} \rangle.$$

The group $W = W(\Upsilon)$ defined by this presentation is called a *Coxeter group*. Classic examples are reflection groups. Let $K = K(\Upsilon)$ be the 2-complex associated with it.

LOT's of Coxeter type

Let Γ be a LOT with vertex set $\mathbf{x}$. We say Γ is of *Coxeter type* if for every edge $e = (x \overset{w}{\to} y)$, the word w contains letters $z \neq x, y$ with even (positive or negative) exponent.

Example. The LOT

$$\Gamma\colon a \overset{bac^2}{\to} b \overset{cba^2}{\to} c$$

is of Coxeter type. Note that the relation $a(bac^2) = (bac^2)b$ reduces to $aba = bab$ modulo squares a^2, b^2, c^2 and gives the Coxeter relation $(ab)^3$. The Coxeter tree associated with Γ is

$$\Upsilon\colon a \overset{3}{--} b \overset{3}{--} c.$$

We have an epimorphism $G(\Gamma) \to W(\Upsilon)$

Theorem. Let Υ be a Coxeter graph on vertices $\mathbf{x}$ such that $W(\Upsilon)_{ab} = \mathbb{Z}_2$. Then there exists a LOT of Coxeter type Γ on vertices $\mathbf{x}$ so that

$$G(\Gamma) \to W(\Upsilon)$$

sending $x \to x$, defines a group epimorphism.

Proof. If $a \stackrel{m}{--} b$ is an edge in Υ_0 (a maximal tree in Υ with odd labels) then make the corresponding edge in Γ to be $a \stackrel{(ba)^{\frac{m-1}{2}}}{\to} b$. □

If you want a prime LOT then fatten up the edge word with even powers of the other letters. For example, in case $m = 3$ instead of

$$w = ba$$

you could take

$$w = b^3 c^{-6} a^5 c^2 d^{20} c^{-2} d^{-12}.$$

Rank of LOT groups

Theorem (Carette-Weidmann 2011). Let Υ be a Coxeter graph with n vertices and assume that all the $m_e \geq 6 \cdot 2^n$. Then the rank of $W(\Upsilon)$ is n.

Corollary. For every n there exist (prime) LOT's of Coxeter type Γ with n vertices such that the rank of $G(\Gamma)$ is n.

Let Γ: $a \stackrel{bac^2}{\to} b \stackrel{cba^2}{\to} c$. Then $rk(G(\Gamma)) = 3$. We don't need the above theorem, only that the group $D_3 *_{\mathbb{Z}_2} D_3$ can not be generated by 2 elements.

Coxeter complex

Let Υ be a Coxeter graph on vertices $\mathbf{x}$ with Coxeter presentation

$$P(\Upsilon) = \langle \mathbf{x} \mid x^2, \in \mathbf{x}, (xy)^{m_e} \text{ if } e = \{x, y\} \text{ is an edge} \rangle.$$

Let $W = W(\Upsilon)$ and $K = K(\Upsilon)$ be the 2-complex associated with it.

Consider the universal covering $\tilde{K}(\Upsilon)$. The 1-skeleton of $\tilde{K}(\Upsilon)$ is the Cayley graph for $(W, \mathbf{x})$. All edges in $\tilde{K}(\Upsilon)$ are double edges. Note that a double edge pair bounds a 2-cell in $\tilde{K}(\Upsilon)$, coming from the relator x^2. We collapse such 2-cells and thus identify each double edge into a single unoriented edge. Every relator $(xy)^{m_e}$ gives rise to $2m_e$ 2-cells with the same boundary. We remove all but one from this set. The 2-complex obtained in this fashion is (the 2-skeleton) of the coxeter $\Sigma(\Upsilon)$.

When Υ is a tree

In case Υ is a tree, $W(\Upsilon)$ is an amalgamated product of dihedral groups and $\Sigma(\Upsilon)$ is a tree of Coxeter cells, which is a thickening of the Bass-Serre tree for the amalgamated product decomposition of $W(\Upsilon)$.

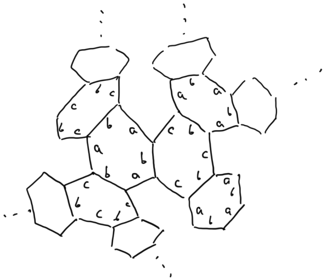

Figure: $\Sigma(\Upsilon)$ in case Υ: $a \overset{3}{--} b \overset{3}{--} c$. The 2 Coxeter relators are $(ab)^3$ and $(bc)^3$.

Let Γ be a labeled oriented tree of Coxeter type and let Υ be the related Coxeter graph. Let $\bar{K}(\Gamma)$ be the normal covering space with fundamental group the kernel of the epimorphism $G(\Gamma) \to W(\Upsilon)$. We will analyze the structure of $\bar{K}(\Gamma)$.

We have maps

$$\bar{K}(\Gamma) \to \tilde{K}(\Upsilon) \to \Sigma(\Upsilon),$$

and note that $\bar{K}(\Gamma)$ and $\tilde{K}(\Upsilon)$ have the same 1-skeleton. If e is an edge in Γ (and hence in Υ) we denote by $\bar{K}_e$ the subcomplex of $\bar{K}(\Gamma)$ that carries the 2-cells with boundary words r_e.

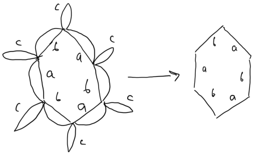

Figure: The 1-skeleton of $\bar{K}_e$ mapping to the corresponding Coxeter cell. Here the LOT edge $e = a \overset{bac^2}{\to} b$ gives the Coxeter edge $a \overset{3}{-\!-} b$.

$\Sigma(\Upsilon)$ is a tree of 2-cells $s\kappa_e$, $e \in \Upsilon$, $s \in W(\Upsilon)$. It follows that $\bar{K}(\Gamma)$ is a tree of 2-complexes $s\bar{K}_e$, $e \in \Gamma$, $s \in W(\Upsilon)$.

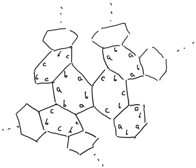

Figure: $\bar{K}(\Gamma)$ "looks like" $\Sigma(\Upsilon)$

To show that $\bar{K}(\Gamma)$ as aspherical you have to show that each $\bar{K}_e$ is aspherical, and that intersections are π_1-injective.

Theorem. $\bar{K}_e$ is aspherical.

Proof. The dihedral group D_{m_e} acts freely on $\bar{K}_e$ and the quotient has a single 2-cell. □

Suppose $e = (a \xrightarrow{w} b)$ is an edge in Γ. Then an *a-side* of $\bar{K}_e$ is a double edge the 1-skeleton with label a, together with all the double edges connected to the two vertices of the a-double edge. A b-side is defined in an analogous way.

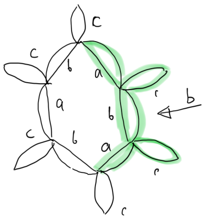

Figure: A b-side.

We say $\bar{K}_e$ is *side injective* if the inclusion $S \to \bar{K}_e$ is π_1-injective for every side. Side injectivity is a question about 1-relator groups.

Theorem. Let Γ be a LOT of Coxeter type. Suppose that $\bar{K}_e$ is side injective for all edges e of Γ. Then $K(\Gamma)$ is aspherical.

A labeled oriented tree Γ is called *label separated* if for every pair of edges e_1 and e_2 that have a vertex in common the intersection $\mathbf{x_{e_1}} \cap \mathbf{x_{e_2}}$ is a proper subset of both $\mathbf{x_{e_1}}$ and $\mathbf{x_{e_2}}$.

Theorem. Let Γ be a label separated LOT of Coxeter type. Then $K(\Gamma)$ is aspherical.

Proof. Freiheitssatz for 1-relator groups. □

Side injectivity

Let $P = \langle a, b, \mathbf{c} \mid r \rangle$, be a 1-relator group, where $\mathbf{c}$ is a finite set of letters (which could be empty). We assume that r is cyclically reduced and contains all generators. We say P is of dihedral m-type, $m \geq 3$, if r reduces to $(ab)^m$ using $a^2 = b^2 = 1$ and free reductions. Let $Q = \langle a, b, \mathbf{c} \mid (ab)^m, a^2, b^2, c \in \mathbf{c} \rangle$. We have an epimorphism $\phi\colon G(P) \to G(Q) = D_m$. Let $\bar{K}(P)$ be the covering of $K(P)$ associated with the kernel. We say P is side injective if the inclusion of a side $S \to \bar{K}$ is π_1-injective. Note that $\bar{K}(P)^{(1)} = \tilde{K}(Q)^{(1)}$, which is a 2m-gon, consisting of double edges labeled alternatingly with a and b, and at every vertex we have a c loop, for every $c \in \mathbf{c}$.

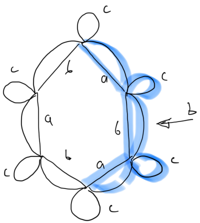

Figure: A b-side in $\bar{K}(P)$, where P is a 1-relator presentation of dihedral 3-type.

Lemma. P is side injective if and only if every cyclically reduced word w that represents the trivial element in $G(P)$ contains a subword s which is a cyclic permutation of

$$a^{\alpha_1} d_1 b^{\beta_1} d_2 a^{\alpha_2} d_3 b^{\beta_2}$$

or it's inverse. The α_i and β_i are odd integers and the d_i are words in the generators containing a and b with even exponents (the d_i could be trivial).

Example

$P = \langle a, b \mid (ab)^m \rangle$, $m \geq 3$, is side injective. This is because 1-relator presentations with torsion are Dehn presentations (in particular $G(P)$ hyperbolic). A word w that is trivial in the group contains a subword of length more that $1/2$ of the relator, hence it contains a subword $a^{\epsilon_1} b^{\epsilon_1} a^{\epsilon_3} b^{\epsilon_4}$, where the $\epsilon_i = \pm 1$.

Example

$P = \langle a, b \mid abab..a = baba...b \rangle$, where the alternating word on the left and the right side of the equation has length $m \geq 5$. Then P is side injective.

Note that $G(P)$ is not hyperbolic because it contains $\mathbb{Z} \times \mathbb{Z}$ subgroups. Let x be the word on the right, then x^2 is a central element. In fact, if we set $y = ba$, then $G(P)$ is the group defined by $\langle x, y \mid x^2 = y^m \rangle$. The quotient $G(P)/\langle x \rangle$ is hyperbolic (it is the free product $\mathbb{Z}_2 * \mathbb{Z}_m$). In fact, $G(P)/\langle x \rangle$ is defined by $\langle a, b \mid (abab...b)^2, (ab)^m \rangle$, which can be shown to be a Dehn presentation. Side injectivity follows from the Lemma. Note that in case $m = 3$ the relation $(aba)^2 = aba^2ba$ contains the subword aba^2, which is more that half of the relator and does fit into a side. So hyperbolicity does not imply side injectivity in case $m = 3$.

Example

Let $P = \langle a, b, \mathbf{c} \mid a(bauba) = (bauba)b \rangle$, where u is a word that does contain some $c \in \mathbf{c}$. Then P is side injective. This follows from the next theorem.

Theorem. Suppose $P = \langle a, b, \mathbf{c} \mid a(u_1 c^{\epsilon_1} u_2 c^{\epsilon_2} u_3) = (u_1 c^{\epsilon_1} u_2 c^{\epsilon_2} u_3)b \rangle$, where

1. $c \in \mathbf{c}$, $\epsilon_i = \pm 1$;
2. the words u_1 and u_3 do not contain c;
3. both $u_1^{-1}a$ and $u_3 b^{-1}$ contain a subword s as in the Lemma .

Then P is side injective.

What if side injectivity fails?

Theorem. Suppose Γ is a LOT of Coxeter type and there exist two edges e_1 and e_2 in Γ so that

1. $\bar{K}_{e_1} \cap \bar{K}_{e_2} = S$;
2. neither $\bar{K}_{e_1}$ nor $\bar{K}_{e_2}$ is side injective, and in fact

$$\ker(\pi_1(S) \to \pi_1(\bar{K}_{e_1})) \cap \ker(\pi_1(S) \to \pi_1(\bar{K}_{e_2})) \neq 1.$$

Then Whitehead's asphericity conjecture is false.