

Combinatorial Relative Asphericity

Jens Harlander and Stephan Rosebrock

July 30, 2020

Abstract

Relative notions of combinatorial asphericity have been used to prove that injective labeled oriented trees (which encode spines of ribbon 2-knots) are aspherical. This article presents an overview and comparison of the different notions of relative combinatorial asphericity. It also contains new results concerning characterizations of relative DR and tests that imply relative combinatorial asphericity. The last section of the article is devoted to examples that illustrate the concepts and the use of the tests given.

Keywords: Diagrammatic reducibility, asphericity, 2-complexes, group-presentations, weight test

MSC 2010: 57M20, 57M05, 57M35, 20F06

1 Introduction

In [10] Huck and the second author showed that prime injective labeled oriented trees (which encode spines of ribbon 2-knots) are aspherical. We realized that in order to extend this result to all injective labeled oriented trees we needed notions of combinatorial asphericity relative to a sub-LOT. Relative vertex asphericity (VA) is used in [7] to prove that injective labeled oriented trees are aspherical. A simpler proof that gives a stronger result is presented in [9].

The purpose of this article is to provide an overview and comparison of the different notions of combinatorial asphericity relative to a subcomplex. Given a pair of 2-complexes $K \subseteq L$, we say that L is diagrammatically reducible (DR) relative to K if every spherical diagram over L can be reduced to a spherical diagram over K by folds across edges with 2-cells from $L - K$. If folds across vertices are also allowed then this gives vertex asphericity (VA) relative to K . Precise definitions can be found in Section 2. Either notion implies that all second homotopy of L is concentrated in K : The inclusion induced map $\pi_2(K) \rightarrow \pi_2(L)$ is surjective. If in addition one also wants π_1 -injectivity $\pi_1(K) \rightarrow \pi_1(L)$ one needs a stronger version of relative combinatorial asphericity, a concept we called directed DR away from K .

Bogley and Pride defined relative presentations $\langle H, \mathbf{x} \mid \mathbf{r} \rangle$, where H is a group and \mathbf{r} is a set of words in $F(\mathbf{x}) * H$. See [2] and [3]. They also define diagrammatic reducibility for relative presentations. Relative presentations have a more group theoretic flavor, but it turns out that DR for relative presentations is closely related to our concept of directed DR.

The paper is organized as follows. In Section 2 we present the different notions of relative combinatorial asphericity. In Section 3 we explain how the various notions relate to each other. In Section 4 we prove a characterization of relative diagrammatic reducibility along the lines of Corson-Trace. In Section 5 we present weight tests and max/min results with which one can show relative combinatorial asphericity. Results that are already published elsewhere are presented without proofs. This section contains new results that have not been published. The last Section 6 is devoted to examples that illustrate the concepts and the use of the tests given in Section 5. Most examples are from the class of labeled oriented trees.

2 Notions of Combinatorial Relative Asphericity

A map $f: X \rightarrow Y$ between complexes is *combinatorial* if f maps open cells of X homeomorphically to open cells of Y . A *surface diagram* over a 2-complex K is a combinatorial map $f: C \rightarrow K$, where C is a surface with a cell structure. If C is a 2-sphere we call f a *spherical diagram*. Note that if we orient the cells in C and label each cell c of C by $f(c)$, the labeling on C carries all information of f . We refer to such a labeled 2-sphere also as a spherical diagram over K . If K is non-aspherical, then there exists a spherical diagram which realizes a nontrivial element of $\pi_2(K)$. In fact, $\pi_2(K)$ is generated by spherical diagrams. So in order to check whether a 2-complex is aspherical or not it is enough to check spherical diagrams.

Let K be a 2-complex. The link of a vertex v , $\text{lk}(K, v)$, is the boundary of a regular neighborhood of v in K . So $\text{lk}(K, v)$ is a graph whose edges are the corners of 2-cells at v . Suppose K is a standard 2-complex with a single vertex v and oriented edge set X . Then the vertices of $\text{lk}(K, v) = \text{lk}(K)$ are $\{x^+, x^- \mid x \in X\}$, where x^+ is a point of the oriented edge x close to the beginning, and x^- is a point close to the ending of that edge. The *positive link* $\text{lk}^+(K)$ is the *full subgraph* on the vertex set $\{x^+ \mid x \in X\}$ and the *negative link* $\text{lk}^-(K)$ is the full subgraph on the vertex set $\{x^- \mid x \in X\}$. Let $f: S \rightarrow K$ be a surface diagram and $v \in S$ a vertex. Restricting to the link we obtain a combinatorial map $f|_{\text{lk}(S, v)}: \text{lk}(S, v) \rightarrow \text{lk}(K)$ for every vertex $v \in S$ and we let $z(v) = c_1 \dots c_q$ be the image, which is a *cycle* (a closed edge path) in $\text{lk}(K)$.

Definition 2.1 Let Γ be a graph and Γ_0 be a subgraph. Let $z = e_1 \dots e_q$ be a cycle. We say

1. z is homology reducible if it contains a pair of edges e_i, e_j such that $e_i = \bar{e}_j$ (the bar indicates opposite orientation) and homology reduced otherwise. z is called reducible if in addition $j = i + 1, \pmod{q}$.
2. z is homology reducible relative to Γ_0 if there is pair of edges e_i, e_j such that $e_i = \bar{e}_j$ which is contained in $\Gamma - \Gamma_0$ and homology reduced relative to Γ_0 otherwise. z is called reducible relative to Γ_0 if in addition $j = i + 1, \pmod{q}$.

Let $f: C \rightarrow L$ be a spherical diagram. A vertex $v \in C$ is called a *folding vertex* if $z(v) = c_1 \dots c_q \in \text{lk}(L)$ is homology reducible. In that case the pair of 2-cells (d_i, d_j) of C containing the preimages of c_i and c_j , respectively, satisfying $c_i = \bar{c}_j$ is called a *folding pair*. If $z(v)$ is reducible, then $j = i + 1$ and d_i and d_j share a common edge in C which is called a *folding edge*.

f is called *vertex reduced* if it does not have a folding vertex and f is called *reduced* if it does not have a folding edge. A 2-complex L is called *vertex aspherical* (VA) if each spherical diagram over L has a folding vertex. L is called *diagrammatically reducible* (DR) if each spherical diagram over L has a folding edge. The vertices in the boundary of a folding edge are folding vertices so DR implies VA. Certainly VA implies asphericity.

Definition 2.2 Let K be a subcomplex of the 2-complex L . We say that

- L is VA relative to K if every spherical diagram $f: C \rightarrow L, f(C) \not\subseteq K$, has a folding vertex with folding pair of 2-cells in $L - K$.
- L is DR relative to K if every spherical diagram $f: C \rightarrow L, f(C) \not\subseteq K$, has a folding edge with folding pair of 2-cells in $L - K$.

Theorem 2.3 Let L be a 2-complex and K a subcomplex. If K is VA and L is VA relative to K then L is VA.

Proof. Assume $f: C \rightarrow L$ is a vertex reduced spherical diagram. Since L is VA relative to K we have that $f(C) \subseteq K$. So $f: C \rightarrow K$ is a vertex reduced spherical diagram, contradicting the assumption that K is VA. \square

Theorem 2.4 Let L be a 2-complex and K a subcomplex. If K is DR and L is DR relative to K then L is DR.

Proof. Assume $f: C \rightarrow L$ is a reduced spherical diagram. Since L is DR relative to K we have that $f(C) \subseteq K$. So $f: C \rightarrow K$ is a reduced spherical diagram, contradicting the assumption that K is DR. \square

It is clear that if L is DR relative to K then L is VA relative to K .

Theorem 2.5 *If L is VA relative to K , then $\pi_2(L)$ is generated, as $\pi_1(L)$ -module, by the image of $\pi_2(K)$ under the map induced by inclusion. In particular, if K is aspherical, then so is L .*

Proof. Every vertex reduced spherical diagram $f: C \rightarrow L$ has its image $f(C)$ in K . Thus f represents an element in $\pi_2(K)$. Since $\pi_2(L)$ is generated by vertex reduced spherical diagrams, it follows that $\pi_2(L)$ is generated by the image of $\pi_2(K)$. \square

There is another notion of relative combinatorial reducibility which is studied in [8] by the authors. For a set X call a subset Y *proper* if $Y \neq X$ (Y may be empty). Remember that if K is a 2-complex with edge set X and if $f: C \rightarrow K$ is a spherical diagram then an edge $e \in C$ is labeled by $x \in X$, if $f(e) = x$.

Definition 2.6 *Let K be a 2-complex with edge set X . Let Y be a proper subset of X . We say that K is*

- DR directed away from Y if every spherical diagram $f: C \rightarrow K$ that contains an edge with label from $X - Y$ also contains a folding edge with label from $X - Y$;
- DR in all directions if every spherical diagram $f: C \rightarrow K$ that contains an edge labeled $x \in X$ also contains a folding edge with label x . Note that this implies that K is DR directed away from all proper $Y \subset X$.

If $Y = \emptyset$ then DR directed away from Y simply means DR. In [8] directed DR is defined via presentations. If the 2-complex K of Definition 2.6 is the standard 2-complex given by a finite presentation then Definition 2.6 is the same as the one in [8] given here: If $P = \langle X \mid R \rangle$ is a presentation and $Y \subset X$ is proper we also say that P is DR directed away from Y meaning that the standard 2-complex $K(P)$ build from P is DR directed away from the 1-cells corresponding to Y .

Let K be a 2-complex with edge set X and let Y be a proper subset of X . We define K_Y to be the subcomplex of K with edge set Y containing exactly those 2-cells of K with all boundary 1-cells from Y . The following theorem is Theorem 2.2 of [8]. We include a proof for the convenience of the reader:

Theorem 2.7 *Let K be a 2-complex with edge set X . Suppose that K is DR directed away from the proper subset $Y \subset X$. Then*

1. $\pi_2(K)$ is generated (as a $\pi_1(K)$ -module) by the image of the inclusion induced map $\pi_2(K_Y) \rightarrow \pi_2(K)$; furthermore
2. every disc diagram $g: D \rightarrow K$ with boundary labeled by a word in Y , that contains a label from $X - Y$, has a folding edge with label from $X - Y$. Consequently, the inclusion induced map $\pi_1(K_Y) \rightarrow \pi_1(K)$ is injective.

Proof. Suppose $f: C \rightarrow K$ is a reduced spherical diagram. If $f(C)$ is not contained in K_Y then C contains an edge e so that $f(e) \notin Y$. Since we assumed that K is DR directed away from Y it follows that C contains a folding edge e' so that $f(e') \notin Y$, contradicting the assumption that $f: C \rightarrow K$ is reduced. Since $\pi_2(K)$ is generated (as a $\pi_1(K)$ -module) by reduced spherical diagrams, the first statement follows.

Suppose $g: D \rightarrow K$ is a disc diagram as in statement (2). We double D and construct a spherical diagram $g': C = D_1 \cup D_2 \rightarrow K$, where D_1 is mapped by g and D_2 is mapped by $-g$ (an orientation reversion followed by g). Note that C contains an edge with label not in Y . Since K is DR away from Y this spherical diagram contains a folding edge with label not from Y . This folding edge can not occur on $\partial D_1 = \partial D_2$. Thus D_1 or D_2 contain an interior folding edge with label not in Y and hence so does D .

We next show π_1 -injectivity. Suppose w is a word in $Y^{\pm 1}$ that represents a non-trivial element of $\pi_1(K_Y)$ that maps to a trivial element in $\pi_1(K)$. Then there exists a reduced Van Kampen diagram $f: M \rightarrow K$ where the boundary of M is labelled by w and is mapped to K_Y . Here M is a planar simply connected region with a cell structure. Note that M is a tree with discs attached at some vertices. One of these discs, say \bar{D} , contains an edge e such that $f(e) \notin K_Y$, otherwise $f(M) \subseteq K_Y$ which would imply that $w = 1$ in $\pi_1(K_Y)$. Thus $\bar{g} = f|_{\bar{D}}: \bar{D} \rightarrow K$ is a disc diagram as in statement (2) and hence contains a folding edge, contradicting the fact that we assumed $f: M \rightarrow K$ is reduced. \square

We end this section with a brief mention of relative presentations due to Bogley and Pride [2]. A *relative presentation* $\hat{P} = \langle H, \mathbf{x} \mid \hat{\mathbf{r}} \rangle$ consists of a group H , a generating set \mathbf{x} and relator set $\hat{\mathbf{r}} \subseteq H * F(\mathbf{x})$ ($F(\mathbf{x})$ is the free group generated by \mathbf{x}). Bogley and Pride defined diagrammatic reducibility for relative presentations in terms of pictures rather than diagrams. Pictures and diagrams are dual concepts. Here is the idea in terms of marked diagrams. Let \bar{P} be the presentation obtained from \hat{P} by forgetting all the H -information. That is $\bar{P} = \langle \mathbf{x} \mid \bar{\mathbf{r}} \rangle$ where $\bar{r} \in \bar{\mathbf{r}}$ is the image of $\hat{r} \in \hat{\mathbf{r}}$ under the projection $H * F(\mathbf{x}) \rightarrow F(\mathbf{x})$. The 2-complex $K(\hat{P})$ is the 2-complex

$K(\bar{P})$ with corners marked by elements from H : If $\hat{r} = x_1 h_1 x_2 h_2 \dots x_t h_t \in \hat{\mathbf{r}}$, where $x_i \in \mathbf{x}^{\pm 1}$ and $h_i \in H$, then $\bar{r} = x_1 \dots x_t$ and we mark the oriented corner from the i -th edge to the $(i + 1)$ -th edge in the boundary of the 2-cell $\Delta(\bar{r})$ (oriented clockwise) by h_i . An example is given in Figure 1.

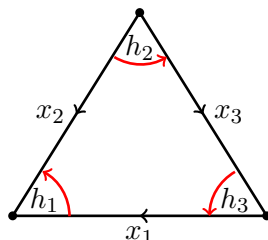


Figure 1: The relator \bar{r} for $\hat{r} = x_1 h_1 x_2^{-1} h_2 x_3 h_3$

In a surface diagram $\hat{f}: \hat{F} \rightarrow K(\hat{P})$ the corners in the 2-cells are marked by elements from H . The corner marking is obtained by pulling back the corner marking of the cells of $K(\hat{P})$. We can assign group elements $h(v)$ to the vertices v in the diagram in the following way: If $c_1(v) \dots c_l(v)$ is the clockwise corner cycle (or path, in case v is a boundary vertex) at the vertex v , and $c_i(v)$ is marked with h_i , then $h(v) = h_1 \dots h_l$. Note that $h(v)$ is defined only up to cyclic permutation in case v is an interior vertex.

A spherical diagram $\hat{f}: \hat{C} \rightarrow K(\hat{P})$ is *admissible* if $h(v) = 1$ for all but possibly one vertex v_0 of \hat{C} . A relative presentation is defined to be *diagrammatically reducible DR* if every admissible spherical diagram has a folding edge.

DR for relative presentations is important in the study of equations over groups. It is known that if \hat{P} is DR, then the map $H \rightarrow G(\hat{P})$ is injective, that is the set of equations $\hat{\mathbf{r}} = \mathbf{1}$ over H has a solution in an overgroup. For a good overview on the topic of relative presentation we refer to [3].

3 Comparing the notions of relative asphericity

Proposition 3.1 *Let K be a 2-complex with edge set X . Let Y be a proper subset of X . If K is DR directed away from Y , then K is DR relative to K_Y .*

Proof. Assume K is not DR relative to K_Y . Then there exists a spherical diagram $f: C \rightarrow K$ such that $f(C) \not\subseteq K_Y$ where all pairs of 2-cells which may be reduced lie in K_Y . Since $f(C) \not\subseteq K_Y$ we have that C contains an edge labelled by an element of $X - Y$. Since all pairs of 2-cells which may be reduced lie in K_Y we have that all folding edges of C are labelled by elements of Y . So K is not DR directed away from Y . \square

If K is DR relative to K_Y then K does not have to be DR directed away from Y . This is because a pair of cancelling 2-cells in a spherical diagram might map to $K - K_Y$ but the common edge of these 2-cells maps to Y .

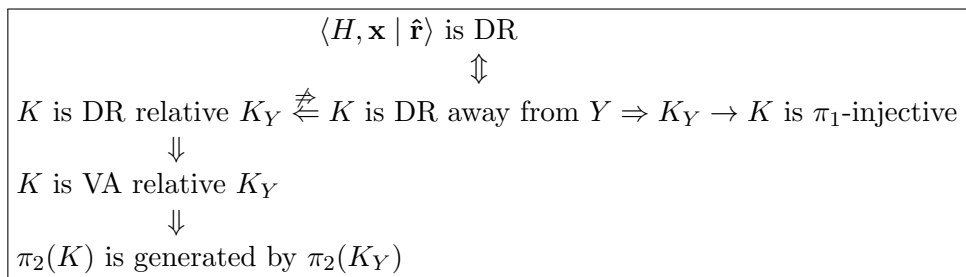
Example 3.2 Let $P = \langle a, b, c \mid bac^{-1}, cb^{-1}a^{-1} \rangle$ be a presentation. $K(P)$ is the torus. There is a disk diagram D with boundary reading $aba^{-1}b^{-1}$ achieved by gluing the two relator disks along c together. Glue D to $-D$ to obtain a spherical diagram which is reducible at a and b only. This shows that P is not DR away from $Y = \{a, b\}$.

Since K_Y is the 2-complex modeled on the presentation $\langle a, b \mid \rangle$ which has no relators we have that $K(P)$ is DR relative K_Y if and only if $K(P)$ is DR. But this presentation of the torus is certainly DR, so $K(P)$ is DR relative K_Y .

Let K be the standard 2-complex given by the presentation $P = \langle X \mid R \rangle$ and let $Y \subseteq X$ proper. We can associate to this a relative presentation: Let $H = \pi_1(K_Y)$, $\mathbf{x} = X - Y$ and $\hat{\mathbf{r}}$ be the set obtained from R in the following way: We have a homomorphism $\phi: F(X) \rightarrow H * F(\mathbf{x})$ by sending $x \in Y$ to the group element in H that it presents, and x to x if $x \in X - Y$. Let $\hat{r} = \phi(r)$, $r \in R$. Then we have a relative presentation $\hat{P} = \langle H, \mathbf{x} \mid \hat{\mathbf{r}} \rangle$. The following result, obtained by the authors, can be found in [8].

Theorem 3.3 P is DR directed away from Y if and only if the relative presentation $\hat{P} = \langle H, \mathbf{x} \mid \hat{\mathbf{r}} \rangle$ is DR.

Let K be a 2-complex with edge-set X and $Y \subset X$ a proper subset. Let $H = \pi_1(K_Y)$ and $\mathbf{x} = X - Y$. The following diagram visualizes the connections between the different notions of relative combinatorial asphericity:



4 A Characterization of Relative Diagrammatic Reducibility

For a complex K let \tilde{K} be its universal covering. Let $K^{(1)}$ be the 1-skeleton of K . Corson and Trace [5] have shown:

Theorem 4.1 *The 2-complex K is DR if and only if every finite subcomplex of the universal covering \tilde{K} collapses into $\tilde{K}^{(1)}$.*

The authors have generalized this result to directed DR in [8]. Here is the corresponding result for relative DR.

Theorem 4.2 *Let L be a 2-complex and K a subcomplex. Then L is DR relative to K if and only if every finite subcomplex of \tilde{L} collapses into $p^{-1}(K) \cup \tilde{L}^{(1)}$, where $p: \tilde{L} \rightarrow L$ is the covering projection.*

Recall that an edge in a 2-complex is called *free* if it occurs exactly once in the boundary of exactly one 2-cell. A 2-complex is called *closed* if it does not have a free edge.

Lemma 4.3 *Let K be a finite 2-complex and d be a 2-cell in K . If K is closed then there exists a reduced surface diagram $f: F \rightarrow K$ such that F is closed and orientable and d is contained in $f(F)$.*

This lemma is in Corson-Trace [4], Theorem 2.1, stated without the fixed 2-cell d . The fact that f hits a specified 2-cell will be important in the relative case. A detailed proof of Lemma 4.3 can be found in [8], Lemma 3.2.

Proof (of Theorem 4.2). Assume first that L is DR relative to K . Suppose the claim is false. Among all finite subcomplexes of \tilde{L} that do not collapse into $p^{-1}(K) \cup \tilde{L}^{(1)}$ let M be one with the minimal number of 2-cells. Note that M does not have a free edge, because a collapse could be performed at that free edge to produce a complex with fewer 2-cells, contradicting minimality.

Let \tilde{d} be a 2-cell in M not contained in $p^{-1}(K) \cup \tilde{L}^{(1)}$. It follows from Lemma 4.3 that there exists a reduced surface diagram $\tilde{f}: F \rightarrow M \subseteq \tilde{L}$, where F is a closed orientable surface and \tilde{d} is contained in $\tilde{f}(F)$. Let $f = p \circ \tilde{f}: F \rightarrow L$. Note that $d = p(\tilde{d})$ is a 2-cell in $f(F)$ not contained in K . We now proceed as in the proof of Lemma 2.1 in [5]: Attach Van Kampen diagrams along cutting curves of F to produce a simply connected 2-complex N_0 and combinatorial maps $F \xrightarrow{\alpha_0} N_0 \xrightarrow{\beta_0} L$ such that $\beta_0 \circ \alpha_0 = f$. The Van Kampen diagrams exist because f lifts to \tilde{f} , so every closed curve in F maps to a closed curve in L that is homotopically trivial. Note furthermore that N_0 is the 2-skeleton of a cell decomposition of the 3-sphere S^3 . Let N be a 2-complex with the minimal number of 2-cells satisfying the following conditions:

1. N is a simply connected 2-skeleton of a cell-decomposition of S^3 ;
2. There exist combinatorial maps $F \xrightarrow{\alpha} N \xrightarrow{\beta} L$ such that $\beta \circ \alpha = f$.

Corson-Trace show that the attaching maps of the 3-cells of S^3 utilize all 2-cells of N and each attaching map $\gamma: C \rightarrow N$ results in a reduced spherical diagram $\beta \circ \gamma: C \rightarrow L$. Choose a 3-cell so that $\beta \circ \gamma(C)$ contains d . Then $\beta \circ \gamma: C \rightarrow L$ is a reduced spherical diagram that is not a diagram over K , contradicting the assumption that L is DR relative to K .

For the other direction assume that $f: C \rightarrow L$ is a spherical diagram which is not already a diagram over K . We can lift it to a spherical diagram $\tilde{f}: C \rightarrow \tilde{L}$. Now $\tilde{f}(C) = M$ is a finite subcomplex of \tilde{L} not already contained in $p^{-1}(K) \cup \tilde{L}^{(1)}$. Since M collapses into $p^{-1}(K) \cup \tilde{L}^{(1)}$, it has a free edge \tilde{e} in the boundary of a 2-cell not contained in $p^{-1}(K)$. Any edge $e \in C$ so that $\tilde{f}(e) = \tilde{e}$ is a folding edge. Thus $f: C \rightarrow L$ is not reduced relative to K . \square

The following corollary can be used as a tool to show that a given presentation defines an infinite group.

Corollary 4.4 *Let P be a presentation of a finite group and T a subpresentation. Then $K(P)$ is DR relative to $K(T)$ if and only if $K(P)$ collapses into $K(T)$.*

For completeness we finish this section with the directed DR version of the Corson-Trace result from [8]. Comparing the two theorems shows again the subtle differences between relative DR and directed DR.

Theorem 4.5 *Let K be a 2-complex with edge set X . Let Y be a proper subset of X . Then K is DR directed away from Y if and only if every finite subcomplex of \tilde{K} collapses into $p^{-1}(K_Y) \cup \tilde{K}^{(1)}$, where only edges that map to $x \in X - Y$ under the covering projection are used as collapsing edges.*

5 Methods for Showing Relative Combinatorial Asphericity

We first give two results which use tree-like conditions in the link-graph to show relative combinatorial asphericity.

Let Γ be a graph and $\hat{\Gamma} = \Gamma_1 \cup \dots \cup \Gamma_n$ be a union of disjoint subgraphs. We write $\Gamma/\hat{\Gamma}$ for the graph obtained from Γ by collapsing each Γ_i to a vertex.

Definition 5.1 *Let Γ be a graph and $\hat{\Gamma} = \Gamma_1 \cup \dots \cup \Gamma_n$ be a disjoint union of subgraphs. We say Γ is a forest relative to $\hat{\Gamma}$ if $\Gamma/\hat{\Gamma}$ has no cycles. Γ is called a tree relative to $\hat{\Gamma}$ if in addition Γ is connected.*

A subcomplex K of a 2-complex L is called *full*, if for every 2-cell $d \in L$ where all boundary cells are in K we have $d \in K$. The attaching maps of 2-cells of a 2-complex K are said to have *exponent sum 0* if for a given orientation of the 1-cells of K the attaching map of each 2-cell d satisfies: when traveling along the boundary of d in clockwise direction, one encounters the same number of positive as of negative edges.

The following theorem is shown in [9]:

Theorem 5.2 *Let L be a finite 2-complex with one vertex and $K = K_1 \vee \dots \vee K_n \subseteq L$. We assume the attaching maps of 2-cells in L have exponent sum 0, and the K_i are full. If $lk^+(L)$ is a forest relative to $lk^+(K)$ or $lk^-(L)$ is a forest relative to $lk^-(K)$ then L is VA relative to K . Furthermore, the inclusion induced homomorphism $\pi_1(K_i) \rightarrow \pi_1(L)$ is injective for every $i = 1, \dots, n$.*

The following theorem is proved in [8]:

Theorem 5.3 *Let K be a finite 2-complex with one vertex and edge set X . We assume the attaching maps of 2-cells in K have exponent sum 0. Assume that $lk^+(K)$ or $lk^-(K)$ is a forest. Then for each $x \in X$, K is DR directed away from $\{x\}$.*

If ω is a real valued function on the set of edges of a graph Γ and $z = e_1 \dots e_p$ is an edge path in Γ , we write

$$\omega(z) = \sum_{1 \leq i \leq p} \omega(e_i).$$

ω is called a *weight function*. The following is a weight test for directed diagrammatic reducibility. It is a generalized version of Gerstens weight test (see [6]). Its proof is given in [8].

Theorem 5.4 *Let K be a 2-complex with cyclically reduced attaching maps of 2-cells and edge set $X \cup Y$ with $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_p\}$. Suppose we can assign weights $\omega(e) \geq 0$ to the edges e of $lk(K)$, such that:*

1. *If e connects y_i^ϵ with y_j^δ , ($\epsilon, \delta = \pm$) then $\omega(e) \geq 1$;*
2. *If one of e 's boundary vertices is y_i^+ or y_i^- , then $\omega(e) \geq 1/2$;*
3. *If z is a reduced cycle in $lk(K)$, then $\omega(z) \geq 2$;*
4. *Let d be a 2-cell of length $\kappa(d)$ from K , then $\sum_{c \in d} \omega(c) \leq \kappa(d) - 2$.*

Then K is DR directed away from Y .

There are at least two weight tests which measure relative vertex asphericity. The most general is the following:

Let L be a 2-complex and $K = K_1 \vee \dots \vee K_n$ be a subcomplex of L . Let $W(L, K)$ be $\text{lk}(L)$ where $\text{lk}(K_i)$ is replaced by two vertices k_i^+, k_i^- and exactly one edge e_i connecting these two vertices for each $1 \leq i \leq n$. So the edges of $W(L, K)$ are the corners of 2-cells of L not in a K_i and edges e_1, \dots, e_n . The vertices of $W(L, K)$ are the vertices of $\text{lk}(L)$ without the vertices of all $\text{lk}(K_i)$ but with two additional vertices k_i^+, k_i^- for each K_i .

Theorem 5.5 *Let L be a 2-complex with one vertex and cyclically reduced attaching maps of 2-cells and let $K = K_1 \vee \dots \vee K_n$ be a subcomplex of L such that no edge of K represents the trivial element in $\pi_1(K)$. If there is a weight function ω on the set of edges of $W(L, K)$ which satisfies:*

1. $\sum_i \omega(c_i) \leq q - 2$ if c_1, \dots, c_q are the corners of a 2-cell of $L - K$,
2. if z is a homology reduced cycle in $W(L, K)$ then $\omega(z) \geq 2$,
3. $\omega(e_i) = 0$ for $1 \leq i \leq n$,

then L is VA relative K .

Proof: Assume $f: C \rightarrow L$ is a vertex reduced spherical diagram such that $f(C) \not\subset K$. Pull back the weights of $W(L, K)$ to corners in 2-cells of C which map to $L - K$. Replace each maximal region $d \in C$ which is mapped to a single K_i by a 2-cell d' (we call d' a *replaced 2-cell*) and achieve a new cell decomposition C' of the 2-sphere. If d is homeomorphic to a disk with m holes then there are m arcs in C' bounding d' on both sides. Assign weight 0 to all corners of d' . Assign the weights of C to the corresponding corners of non-replaced 2-cells of C' .

The curvature of those 2-cells of C' coming from 2-cells of C mapped to $L - K$ have curvature less or equal to 0 by condition 1. A replaced 2-cell $d' \in C'$ has at least two corners by the condition that no edge of K represents the trivial element in $\pi_1(K)$. It has weight 0 and condition 1 is satisfied for this 2-cell, leading also to non-positive curvature for replaced 2-cells.

Assume z' is the link of a vertex in C' . If z' contains no corners of replaced 2-cells then it has weight at least two by condition 2 since f is homology reduced. If z' contains a corner of a replaced 2-cell, this corner will contribute 0 to the weight of z' . Since it does not appear in $W(L, K)$ if it is a corner in $\text{lk}^+(K_i)$ or $\text{lk}^-(K_i)$ or it has weight 0 in $W(L, K)$ we have weight at least two by condition 2 for z' .

So we have non-positive curvature at vertices of C' contradicting the Euler-Characteristic of C' by the combinatorial Gauss-Bonnet theorem. \square

Next we present a weight test with stronger requirements than the ones in Theorem 5.5 but on the other hand fits to more presentations satisfying the

requirements. Assume L is a 2-complex and $K = K_1 \vee \dots \vee K_n \subseteq L$. We assume L contains a single vertex v . We define $\text{lk}(L, K)$, the link of v in L relative to K in the following way: If y_1, \dots, y_l are the edges of K_i then we denote by $\Delta(K_i)$ the full graph on the vertices $y_i^{\pm 1}$ of $\text{lk}(K_i)$ together with an edge attached at each y_i^+ (a loop at that vertex) and at each y_i^- . Every pair of vertices in $\Delta(K_i)$ is connected by an edge, and at every vertex we have a loop. For each i we remove $\text{lk}(K_i)$ from $\text{lk}(L)$ and insert $\Delta(K_i)$ instead. The resulting graph is $\text{lk}(L, K)$. The proof of the following theorem is given in [9].

Theorem 5.6 *Assume $K = K_1 \vee \dots \vee K_n \subseteq L$, each K_i is full and the attaching maps of 2-cells of K have exponent sum zero. Assume there is a weight function on the edges of $\text{lk}(L, K)$ satisfying*

1. $\sum_i \omega(c_i) \leq q - 2$ if c_1, \dots, c_q are the corners of a 2-cell of L not contained in K ,
2. if z is a homology reduced cycle in $\text{lk}(L, K)$ containing at least one corner from $\text{lk}(L, K) - \Delta(K)$ then $\omega(z) \geq 2$,
3. $\omega(c) = 0$ if c is an edge of $\Delta^+(K_i)$ or $\Delta^-(K_i)$ and $\omega(c) = 1$ if c connects a vertex of $\Delta^+(K_i)$ with one of $\Delta^-(K_i)$.

Then L is VA relative to K . If in addition the attaching maps of the 2-cells of L have exponent sum zero, then all the inclusion induced homomorphisms $\pi_1(K_i) \rightarrow \pi_1(L)$ are injective.

We recall the Freiheitssatz for 1-relator groups: Suppose $P = \langle x_1, \dots, x_n \mid r \rangle$ is a 1-relator presentation, where r is a cyclically reduced word that contains all the generators. Then any proper subset Y of $\{x_1, \dots, x_n\}$ generates a free subgroup of the corresponding group $G(P)$ with basis Y . The following three results, already contained in [8], can be viewed as multi-relator versions of this celebrated result. For convenience to the reader we include short proofs.

If $P = \langle X \mid R \rangle$ is a presentation and $Y \subset X$ define P_Y to be the subpresentation of P with generators Y and all relators that contain only generators of Y .

Theorem 5.7 *Let $P = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a presentation such that P is DR in all directions. Then the inclusion induced homomorphism $G(P_Y) \rightarrow G(P)$ is injective for every subset Y of the generators.*

Proof. If Y is the set of generators of P then $G(P_Y) = G(P)$ and the statement is true. If Y is a proper subset of the set of generators then the result follows from Theorem 2.7 using the fact that P is DR directed away from Y . \square

Theorem 5.8 *Let $P = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ and Y a proper subset of the generators. Assume that each r_i contains a generator not from Y . If P is DR directed away from Y , then Y generates a free subgroup of $G(P)$ with basis Y .*

Proof. Since each r_i contains a generator not from Y we have that $P_Y = \langle Y \mid \rangle$ and $G(P_Y)$ is free. Now Theorem 2.7 (2) gives the desired result. \square

Corollary 5.9 *Let $P = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a presentation where each r_i contains all the generators. If P is DR in all directions, then any proper subset Y of $\{x_1, \dots, x_n\}$ generates a free subgroup of $G(P)$ with basis Y .*

Here is a strengthening of the classical Freiheitssatz which is proved in [8]:

Theorem 5.10 *Let $P = \langle x_1, \dots, x_n \mid r \rangle$ be a one-relator presentation of a group G where r is a cyclically reduced relator that is not a proper power. Then P is DR in all directions.*

6 Examples

Many examples we present are LOGs. A standard reference for labeled oriented graphs, LOGs for short, is [12]. Here are the basic definitions. A LOG is an oriented finite graph Γ on vertices \mathbf{x} and edges \mathbf{e} , where each oriented edge is labeled by a vertex. Associated with a LOG Γ is the *LOG-complex* $K(\Gamma)$, a 2-complex with a single vertex, edges in correspondence with the vertices of Γ and 2-cells in correspondence with the edges of Γ . The attaching map of a 2-cell d_e is the word $xz(zy)^{-1}$, where e is an edge of Γ starting at x , ending at y , and labeled with z .

A labelled oriented graph is called *compressed* if no edge is labelled with one of its vertices. A LOG Γ is called *boundary reducible* if there is a boundary vertex $x \in \Gamma$ which does not occur as an edge label and *boundary reduced* otherwise. A LOG is *injective* if each vertex occurs as an edge label at most once. An injective LOG is called *reduced* if it is compressed and boundary reduced. A *labeled oriented tree*, LOT, is a labeled oriented graph where the underlying graph is a tree. If Γ is a LOT and Γ_1 is a sub-tree of Γ with at least one edge, such that each edge label of Γ_1 is a vertex of Γ_1 , then we call Γ_1 a *sub-LOT* of Γ .

A main result of [9] is:

Theorem 6.1 *Let Γ be a compressed injective LOT. Then $K(\Gamma)$ is VA.*

The main step used in the proof is the following: If Γ is a reduced injective LOT which contains disjoint sub-LOTS $\Gamma_1, \dots, \Gamma_n$, then in most cases $K(\Gamma)$ is VA relative to $K(\Gamma_1) \cup \dots \cup K(\Gamma_n)$. This is shown by using the weight test of Theorem 5.6. Since by induction the Γ_i are VA the result is shown by Theorem 2.3.

The next lemma gives a tool for proving directed DR in many situations (see [8]). Let $T_1 = \langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle$ and $T_2 = \langle y_1, \dots, y_p \mid s_1, \dots, s_q \rangle$ be presentations. Suppose we have a map $\phi_0: F(x_1, \dots, x_k) \rightarrow F(y_1, \dots, y_p)$ that induces a group homomorphism $G(T_1) \rightarrow G(T_2)$. For $n > k$ we can extend ϕ_0 to $\phi: F(x_1, \dots, x_n) \rightarrow F(y_1, \dots, y_p, x_{k+1}, \dots, x_n)$ by defining $\phi(x_i) = x_i$ for $i > k$. Now let P_1 be a presentation of the form

$$P_1 = \langle x_1, \dots, x_k, x_{k+1}, \dots, x_n \mid r_1, \dots, r_l, r_{l+1}, \dots, r_m \rangle.$$

We assume each relator r_j , $j > l$, contains a generator x_i , $i > k$, so that $P_{Y_1} = T_1$ for $Y_1 = \{x_1, \dots, x_k\}$. Let

$$P_2 = \langle y_1, \dots, y_p, x_{k+1}, \dots, x_n \mid s_1, \dots, s_q, \phi(r_{l+1}), \dots, \phi(r_m) \rangle.$$

We use $\phi: P_1 \rightarrow P_2$ as shorthand for the situation just described.

Lemma 6.2 *Suppose we have $\phi: P_1 \rightarrow P_2$. If P_2 is DR directed away from $Y_2 = \{y_1, \dots, y_p\}$, then P_1 is DR directed away from $Y_1 = \{x_1, \dots, x_k\}$.*

See [8] for a proof. The simplest choice for T_2 is the empty presentation $T_2 = \langle \mid \rangle$, that is $G(T_2)$ is trivial as in the following example.

Example 6.3 Consider

$$P_1 = \langle x_1, \dots, x_k, a, b \mid u_1 a u_2 b u_3 a^{-1} u_4 b^{-1} \rangle \rightarrow P_2 = \langle a, b \mid a b a^{-1} b^{-1} \rangle,$$

where the u_i are words in $x_1^{\pm 1}, \dots, x_k^{\pm 1}$. Since P_2 is DR (directed away from \emptyset), it follows that P_1 is DR directed away from $Y = \{x_1, \dots, x_k\}$. More general: Take any DR presentation P_2 . Add generators $Y = \{x_1, \dots, x_k\}$ and insert words in $Y^{\pm 1}$ into the relators of P_2 and one obtains a presentation P_1 which is DR directed away from Y .

Theorem 5.2 is a strong tool for showing asphericity of LOTs. Here is a simple example:

Example 6.4 Let Γ be the LOT depicted in Figure 2. Let Γ_0 be the sub-LOT consisting of the first two edges between x_1 and x_3 (colored red). $K(\Gamma_0)$ is VA by Theorem 6.1 because Γ_0 is injective. Since $lk^-(K(\Gamma))$ is a forest relative to $lk^-(K(\Gamma_0))$ Theorem 5.2 implies that $K(\Gamma)$ is VA relative to $K(\Gamma_0)$. By Theorem 2.3 $K(\Gamma)$ is VA.

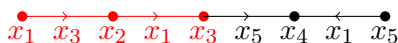


Figure 2: An aspherical LOT Γ with sub-LOT Γ_0 .

Building an example out of an aspherical sub-LOT can even be done in a 2-step (or more steps) process such that the edges you add lead to cycles in the negative and the positive graph:

Example 6.5 Let $T = \langle X \mid R \rangle$ be an arbitrary compressed VA LOT presentation. Assume $x_i, x_j, x_p, x_q \in X$ (not necessarily pairwise distinct). Let $P_1 = \langle X, a, b \mid R, ab = bx_i, ba = ax_j \rangle$. Since $lk^-(K(P_1))$ is a forest relative to $lk^-(K(T))$ Theorem 5.2 implies that $K(P_1)$ is VA relative $K(T)$. Now let

$$P_2 = \langle X, a, b, c, d \mid R, ab = bx_i, ba = ax_j, x_p c = cd, x_q d = dc \rangle$$

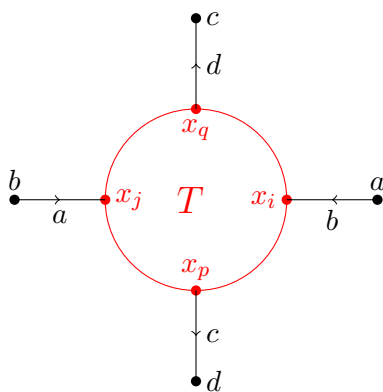


Figure 3: An aspherical LOT build in two steps.

(see Figure 3). Since $lk^+(K(P_2))$ is a forest relative to $lk^+(K(P_1))$, Theorem 5.2 and Theorem 2.3 imply that $K(P_2)$ is a VA LOT-complex.

In this example edges were added to the LOT corresponding to T which have cycles in both, negative and positive graph but still asphericity may be shown by using Theorem 5.2 twice.

Example 6.6 Let $T = \langle X \mid R \rangle$ be an arbitrary compressed VA LOT presentation. Assume $x_i, x_j \in X$ (not necessarily distinct). Let

$$P = \langle X, u, v, w, y \mid R, wy = yu, x_i w = wu, vw = wy, vu = ux_j \rangle$$

(see Figure 4). Since $lk^+(K(P))$ is a forest relative to $lk^+(K(T))$ Theorem 5.2 implies that $K(P)$ is a VA LOT-complex.

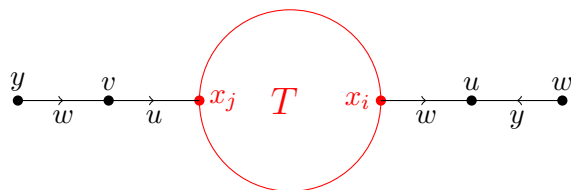


Figure 4: An aspherical LOT.

Let H be the group defined by T and let

$$\hat{P} = \langle H, u, v, w, y \mid wy = yu, x_i w = wu, vw = wy, vu = ux_j \rangle$$

be a relative presentation in the sense of Bogley and Pride (see [2]). In their paper they define a weight test. \hat{P} does not satisfy this weight test if $x_i^3 = x_j^3$ (which is of course satisfied for $x_i = x_j$). This can be seen by drawing the Whitehead graph in the sense of Bogley and Pride and using the simplex method to show that the weight test does not apply.

Example 6.7 Let $T = \langle X \mid R \rangle$ be an arbitrary compressed aspherical LOT presentation. Assume $x_i, x_j \in X$ (not necessarily distinct). Let

$$P = \langle X, u, v, w, y, \mid R, uv = vw, x_i y = yw, vw = wy, vu = ux_j \rangle$$

(see Figure 5).

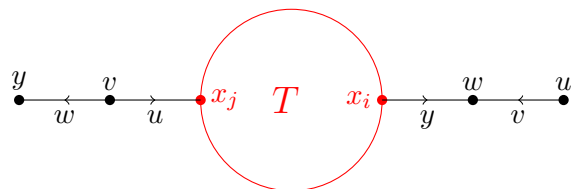


Figure 5: An aspherical LOT.

P and T do not satisfy the conditions of Theorem 5.2 since cycles occur in $lk^+(K(P))$ relative to $lk^+(K(T))$ and $lk^-(K(P))$ relative to $lk^-(K(T))$. There is also no way to build up P in two steps as in Example 6.5. On the other hand it can easily be seen that P satisfies the weight test of Bogley and Pride (see [2]). Give all edges which occur twice in $lk(K(P - T))$ weight 1 and all other edges weight 0. P satisfies the relative weight test of Theorem 5.6 since it is injective relative to $T = \langle X \mid R \rangle$. See page 11, after Lemma 5 in [7] for the definition of “injective relative to”.

The small cancellation conditions $C(p)$, $T(q)$ are defined for instance in the book of Lyndon and Schupp [11]. The following theorem is shown in [8]. It is an application of the weight test (Theorem 5.4):

Theorem 6.8 *Let P be a finite presentation with cyclically reduced relators and Y a subset of the generators. Assume that P is $C(4)$, $T(4)$ or $C(6)$, $T(3)$ and that no two consecutive letters in a (cyclically read) relator of P are elements of $Y^{\pm 1}$. Then P is DR directed away from Y .*

In Rosebrock [13] it is described how to check whether a LOT is $C(4)$, $T(4)$. If a LOT is $C(4)$, $T(4)$ then Theorem 6.8 implies that it is DR away from any of its generators. But there is more in concrete cases:

Example 6.9 Consider the LOT Γ of Figure 6 with any orientation of its edges. This LOT is $C(4)$, $T(4)$. If you choose Y to be the edges of $K(\Gamma)$ corresponding to one of the sets

$$\{x_1, x_2, x_5\}, \{x_1, x_2\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_2, x_5\}, \{x_2, x_6\}, \{x_2, x_7\}, \{x_3, x_6\}, \\ \{x_3, x_7\}, \{x_4, x_7\}$$

then Theorem 6.8 implies that $K(\Gamma)$ is DR away from Y .

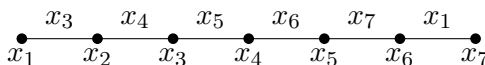


Figure 6: A labelled oriented tree which is $C(4)$, $T(4)$.

Let Γ be a compressed LOT and Γ_0 be a maximal proper sub-LOT with vertex set Y . Let $y \in \Gamma_0$ be any vertex. Let $\bar{\Gamma}$ be the LOT obtained from Γ by collapsing all of Γ_0 to the vertex y . Every occurrence of a vertex x of Γ_0 in $\Gamma - \Gamma_0$ is replaced by y in $\bar{\Gamma}$. The following theorem is shown in [8], it is an application of Lemma 6.2, Theorem 5.3 and Theorem 6.8:

Theorem 6.10 *If $\bar{\Gamma}$ is compressed and $K(\bar{\Gamma})$ is DR directed away from y then Γ is DR directed away from Y , the edges of Γ_0 . In particular if*

1. $lk^+(K(\bar{\Gamma}))$, or $lk^-(K(\bar{\Gamma}))$ is a tree, or
2. $lk(K(\bar{\Gamma}))$ does not contain cycles of length less than four,

then Γ is DR directed away from Y .

Example 6.11 Figure 7 shows a compressed LOT Γ together with a sub-LOT Γ_0 with vertices $Y = \{x_1, \dots, x_5\}$. Below Γ we see the LOT $\bar{\Gamma}$ obtained from Γ by collapsing Γ_0 to the vertex y .

Note that $lk^+(K(\bar{\Gamma}))$ is a tree, so by Theorem 6.10 Γ is DR directed away from Y . Also observe that $lk^-(K(\Gamma_0))$ is a tree, so $K(\Gamma_0)$ is DR, which implies that $K(\Gamma)$ is aspherical by Theorem 2.7 (1). Note that neither

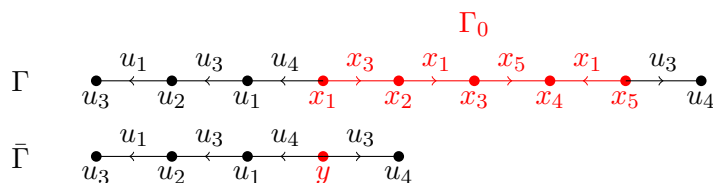


Figure 7: The LOT Γ with sub-LOT Γ_0 and the LOT $\bar{\Gamma}$

$lk^+(K(\Gamma))$ nor $lk^-(K(\Gamma))$ is a tree.

The process of collapsing a sub-LOT in a given labeled oriented tree can also be reversed: If $\bar{\Gamma}$ is a labeled oriented tree, y is a vertex in $\bar{\Gamma}$, and Γ_0 is a labeled oriented tree, we remove y from $\bar{\Gamma}$ and insert Γ_0 to obtain a labeled oriented tree Γ that contains Γ_0 . Collapsing Γ_0 in Γ to a vertex y brings us back to $\bar{\Gamma}$. So the previous theorem can also be stated as follows: If $\bar{\Gamma}$ is a labeled oriented tree that satisfies either condition (1) or (2) of Theorem 6.10, then inserting any LOT Γ_0 into $\bar{\Gamma}$ results in a labeled oriented tree Γ for which $K(\Gamma)$ is DR directed away from the set Y of edges of $K(\Gamma_0)$.

LOTs Γ where $lk^+(K(\Gamma))$ or $lk^-(K(\Gamma))$ is a tree abound. If a LOT Γ' is obtained from a LOT Γ by changing some edge orientations, we call Γ' a *reorientation* of Γ . In [10] Proposition 5.1, Huck and Rosebrock show that each LOT Γ has a reorientation Γ' such that $lk^+(K(\Gamma'))$ is a tree. Theorem 5.3 now implies

Theorem 6.12 *Each LOT Γ has a reorientation Γ' so that $K(\Gamma')$ is DR away from any one of its edges.*

In [13] the second author gives conditions on a labeled oriented tree so that condition (2) of Theorem 6.10 holds.

Example 6.13 Figure 8 shows a labelled oriented tree Γ (orientations can be chosen at will) with a sub-LOT Γ_0 between u_4 and u'_4 (which can be filled in at will).

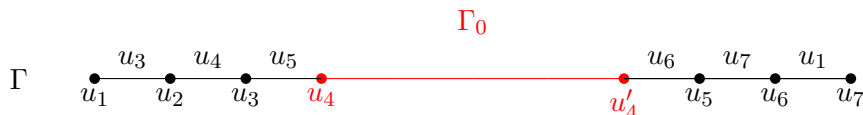


Figure 8: A labeled oriented tree Γ with sub-LOT Γ_0 .

Notice that if we collapse the red sub-LOT Γ_0 to the vertex $y = u_4$, we obtain a labeled oriented tree $\bar{\Gamma}$ for which $lk(K(\bar{\Gamma}))$ does not contain cycles

of length less than four. It follows from Theorem 6.10 that $K(\Gamma)$ is DR away from the edge set of $K(\Gamma_0)$.

At last we provide an example to Theorem 5.5.

Example 6.14 Let $n \geq 2$ and $1 \leq k \leq n$. Let $P_1 = \langle y_1, \dots, y_k \mid R_1 \rangle$ and $P_2 = \langle y_{k+1}, \dots, y_n \mid R_2 \rangle$ and K_i be the standard 2-complex corresponding to P_i . Assume R_i is a set of relators, such that each y_m is nontrivial in $\pi_1(K_i)$ ($i = 1, 2$). Let L be the standard 2-complex given by

$$P = \langle x_1, \dots, x_n, y_1, \dots, y_n \mid R_1, R_2, w_1, \dots, w_n \rangle$$

where $w_i = x_i^{-2} y_i x_{i+1} y_i^{-1}$ ($i \bmod n$). By Theorem 5.5 L is VA relative $K = K_1 \vee K_2$. This can be seen by assigning weight 0 to corners in $lk^+(L-K)$ or $lk^-(L-K)$ and weight 1 for the other corners of $L-K$. Observe that relators have not exponent sum 0.

References

- [1] M. A. Blufstein and E. G. Minian, *Strictly systolic angled complexes and hyperbolicity of one-relator groups*, <https://arxiv.org/abs/1907.06738>, (2019).
- [2] W. Bogley and S. Pride, *Aspherical relative presentations*. Proc. of the Edinburgh Math. Society 35, (1992), pp. 1–39.
- [3] W. Bogley, M. Edjvet and G. Williams, *Aspherical relative presentations all over again*, in: Groups St Andrews 2017 in Birmingham, London Mathematical Society Lecture Note Series, Editors: Campbell, C. M. and Parker, C. W. and Quick, M. R. and Robertson, E. F. and Roney-Dougal, Cambridge University Press (2019), pp. 169-199.
- [4] J. M. Corson and B. Trace, *Geometry and algebra of nonspherical 2-complexes*, J. London Math. Soc. 54, (1996), pp. 180-198.
- [5] J. M. Corson and B. Trace, *Diagrammatically reducible complexes and Haken manifolds*, J. Austral. Math. Soc. (Series A) 69, (2000), pp. 116-126.
- [6] S. M. Gersten, *Reducible diagrams and equations over groups*, in: Essays in Group Theory (S. M. Gersten editor), MSRI Publications 8 (1987), pp. 15–73.
- [7] J. Harlander and S. Rosebrock, *Injective labelled oriented trees are aspherical*, Mathematische Zeitschrift 287 (1), (2017), pp. 199–214.

- [8] J. Harlander and S. Rosebrock, *Directed diagrammatic reducibility*, to appear in *Topology and its Applications* (2020).
- [9] J. Harlander and S. Rosebrock, *Relative Vertex Asphericity*, to appear in: *Canad. Math. Bull.* (2020).
- [10] G. Huck and S. Rosebrock, *Aspherical Labelled Oriented Trees and Knots*, *Proceedings of the Edinburgh Math. Soc.* 44, (2001), pp. 285-294.
- [11] R. Lyndon and P. Schupp. *Combinatorial group theory*, Springer Verlag, Berlin, (1977).
- [12] S. Rosebrock, *Labelled Oriented Trees and the Whitehead Conjecture*; *Advances in Two-Dimensional Homotopy and Combinatorial Group Theory*; Cambridge University Press, LMS Lect. Notes 446, editors W. Metzler, S. Rosebrock; (2018), pp. 72–97.
- [13] S. Rosebrock, *On the realization of Wirtinger presentations as knot groups*, *Journal of Knot Theory and its Ramifications*, Vol. 3 (1994), pp. 211-222.
- [14] A. Sieradski, *A coloring test for asphericity*, *Quart. J. Math. Oxford* (2) 34 (1983), pp. 97–106.

Jens Harlander
Department of Mathematics
Boise State University
Boise, ID 83725-1555
USA
email: jensharlander@boisestate.edu

Stephan Rosebrock
Pädagogische Hochschule Karlsruhe
Bismarckstr. 10
76133 Karlsruhe
Germany
email: rosebrock@ph-karlsruhe.de