Actions of A_5 on contractible 2-complexes

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The conjecture remained open for A_5.

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Corollary (S.C.)

Let π be the fundamental group of a fixed point free, finite and acyclic 2-dimensional A_5-complex. Then π is infinite or there is an epimorphism $\pi \to A_5$.

Steps in the proof

- Using the results of Oliver–Segev [OS02] we reduce the problem to the study of the fundamental group of a 2-complex obtained from the graph $\Gamma_{OS}(A_5)$ attaching $k \geq 0$ free orbits of 1-cells and $k + 1$ free orbits of 2-cells.

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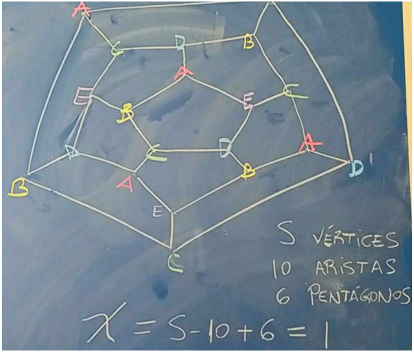
- We attack this problem by constructing a moduli of representations of Γ_k in $\mathrm{SO}(3)$. This argument is inspired by Howie's proof of the Scott–Wiegold conjecture [How02].

Poincaré's dodecahedral space



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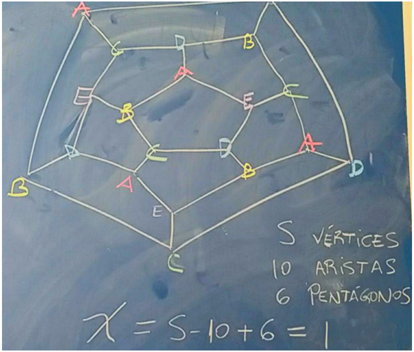

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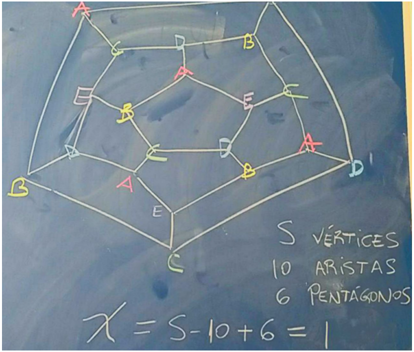

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The graph has three orbits of vertices (with stabilizers A_4, D_6 and D_{10}) and three orbits of edges (with stabilizers C_2, C_2 and C_3).

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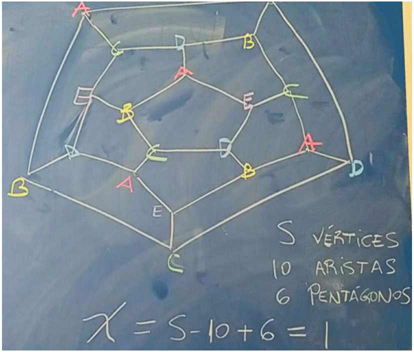

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Attaching a free orbit of 2-cells along a closed edge path of length 3 we recover Poincaré's dodecahedron which is acyclic and has fundamental group of order 120.

2-dimensional acyclic A_5-complexes

Theorem

Let X be a fixed point free 2-dimensional finite acyclic A_5-complex. Then there is a fixed point free 2-dimensional finite acyclic A_5-complex X' obtained from the graph $\Gamma_{OS}(A_5)$ by attaching $k \geq 0$ free orbits of 1-cells and $k+1$ free orbits of 2-cells and an epimorphism $\pi_1(X) \to \pi_1(X')$.

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Brown's short exact sequence

If X is a G-complex we can consider the group extension $\pi_1(X) : G$ obtained by lifting the maps $g\colon X \to X$ to the universal cover $\widetilde{X}$ of X.

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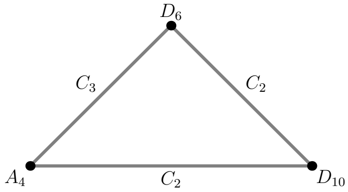
The group written in each vertex (resp. edge) of X/G is the stabilizer of a representative of its orbit.

Applying Brown's result



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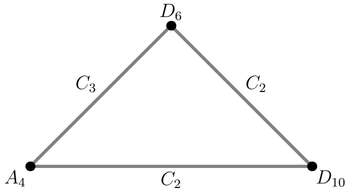

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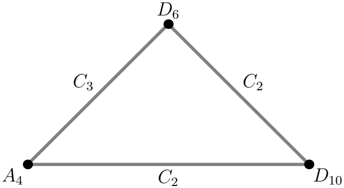

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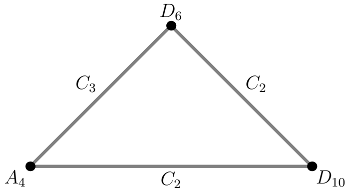

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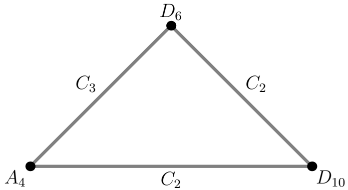

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The following are equivalent.

(i) *Every finite, 2-dimensional contractible A_5-complex has a fixed point.*

(ii) *There is no presentation of A_5 of the form*

$$\langle a, b, c, d, x_0, \dots, x_k \mid a^2, b^3, c^2, d^2, (ab)^3, (bc)^2, (cd)^5, x_0 a x_0^{-1} = d, w_0, \dots, w_k \rangle$$

with $w_0, \dots, w_k \in \ker(\phi)$, where $\phi\colon F(a, b, c, d, x_0, \dots, x_k) \to A_5$ is given by $a \mapsto (2,5)(3,4)$, $b \mapsto (3,5,4)$, $c \mapsto (1,2)(3,5)$, $d \mapsto (2,5)(3,4)$ and $x_i \mapsto 1$.

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The idea is, given words $w_0, \dots, w_k \in N$, to construct a representation ρ of Γ_k such that $\rho(w_i) = 1$ for every i and such that $\rho(u) \neq 1$ for some $u \in \{x_0, \dots, x_k, (bac)^3\}$.

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To do this we choose rotations θ_1, θ_2 θ_3 centralizing $r_1 r_2$, $r_2 r_3$, and $r_3 r_4$ respectively.

A moduli of representations

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$$\Gamma_k = \langle a, b, c, d, x_0, \ldots, x_k \mid a^2, b^3, c^2, d^2, (ab)^3, (bc)^2, (cd)^5, x_0 a x_0^{-1} = d \rangle.$$

in many different ways.

We first construct a single representation $\Gamma_0 \to \mathrm{SO}(3)$ by choosing reflections r_1, r_2, r_3, r_4, r_5 so that $a \mapsto r_1 r_2$, $b \mapsto r_2 r_3$, $c \mapsto r_3 r_4$, $d \mapsto r_4 r_5$ defines a representation of the subgroup generated by a, b, c and d.

Since $r_1 r_2$ are $r_4 r_5$ rotations of angle π they are conjugated by a rotation r_6 and we can extend this to a representation of Γ_0 by mapping $x_0 \mapsto r_6$.

We then twist the representation, giving three degrees of freedom.

To do this we choose rotations θ_1, θ_2 θ_3 centralizing $r_1 r_2$, $r_2 r_3$, and $r_3 r_4$ respectively.

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Finally to represent Γ_k we choose freely the image of the x_i with $i \geq 1$.

A moduli of representations (cont.)

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Each $w \in \Gamma_k$ induces a smooth map W given by $\mathbf{z} \mapsto \rho_{\mathbf{z}}(w)$.

The universal representation $\mathbf{z}_u$

Remark

Given $\{w_i\}_{i\in I} \subseteq F(a,b,c,d,x_0,\dots,x_k)$*, the set of points in* $\mathbb{C}^6 \times \mathrm{SO}(3,\mathbb{C})^k \subseteq \mathbb{C}^{6+9k}$ *such that* $\rho(w_i) = 1$ *for all* $i \in I$ *is an affine algebraic variety* $Z(\{w_i \ : \ i \in I\})$*.*

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Proposition

There is a unique point $(\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3,X_1,\dots,X_k)$ *in* $\mathbb{C}^6 \times \mathrm{SO}(3,\mathbb{C})^k$*, with* $\alpha_i^2+\beta_i^2 = 1$*, such that*

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This unique solution

$$\mathbf{z}_u = (\alpha_1^u,\beta_1^u,\alpha_2^u,\beta_2^u,\alpha_3^u,\beta_3^u,1,\dots,1)$$

is real and it is given by

$$\begin{array}{lll} \alpha_1^u = -\frac{1}{4}\sqrt{3\sqrt{5}+9} & \alpha_2^u = -\sqrt{-\frac{2}{15}\sqrt{5}+\frac{1}{3}} & \alpha_3^u = -\sqrt{-\frac{1}{5}\sqrt{5}+\frac{1}{2}} \\ \beta_1^u = \frac{1}{4}\sqrt{-3\sqrt{5}+7} & \beta_2^u = \sqrt{\frac{2}{15}\sqrt{5}+\frac{2}{3}} & \beta_3^u = \sqrt{\frac{1}{5}\sqrt{5}+\frac{1}{2}} \end{array}$$

Finding representations

Theorem

Let $w_0, \dots, w_k \in \ker(\phi)$ *and* $N = \ker(\overline{\phi}\colon \Gamma_k \to A_5)$*. If* $N = \langle\langle w_0, \dots, w_k \rangle\rangle^{\Gamma_k}[N, N]$ *then the variety* $Z(w_0, \dots, w_k)$ *has at least two points.*

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The hypothesis $N = \langle\langle w_0, \dots, w_k \rangle\rangle^{\Gamma_k}[N, N]$ is equivalent to the acyclicity of the corresponding 2-complex.

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It is also equivalent to $w_0, \ldots, w_k$ being a generating set for the A_5-module $N/[N, N]$ (i.e. the relation module of $1 \to N \to \Gamma_k \xrightarrow{\overline{\phi}} A_5 \to 1$).

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$S^3 = \{q \in \mathbb{H} \,:\, |q| = 1\}$ acts on $S^2 = \{b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \,:\, b^2 + c^2 + d^2 = 1\}$ by conjugation.

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Every element in $S^3 - \{\pm 1\}$ can be written as $\cos(\theta/2) + \sin(\theta/2)q$ with $\theta \in [0, 2\pi]$ and $q = b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in S^2$.

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Let $\varphi\colon \mathbb{D}^3 \to S^3 \subset \mathbb{H}$ be the map $(b, c, d) \mapsto \sqrt{1 - b^2 - c^2 - d^2} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$.

Lifting to S^3

Let $t_1, t_2, t_3, \dots, t_{3(k+1)}$ be the coordinates of $[0, 2\pi]^3 \times (\mathbb{D}^3)^k$.

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$$\begin{aligned}
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\widetilde{C}(\mathbf{t}) &= \widetilde{R}(t_1)\, \widetilde{S}_0\, \widetilde{R}(t_1)^{-1},\\
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\end{aligned}$$

For $i = 1, \ldots, k$ *we define* $\widetilde{X}_i(\mathbf{t}) = \varphi(t_{3i+1}, t_{3i+2}, t_{3i+3})$.

Lifting to S^3

Let $t_1, t_2, t_3, \dots, t_{3(k+1)}$ be the coordinates of $[0, 2\pi]^3 \times (\mathbb{D}^3)^k$.

Definition

Let $\widetilde{A}$, $\widetilde{B}$, $\widetilde{S}_0$, $\widetilde{S}_1$, $\widetilde{S}_2$, $\widetilde{S}_3$, $\widetilde{S}_4$, *be preimages of* A, B, S_0, S_1, S_2, S_3, S_4 *by* p.

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We again have a smooth map $\widetilde{W} \colon [0, 2\pi]^3 \times (\mathbb{D}^3)^k \to (S^3)^{k+1}$ induced by each word $w \in F(a, b, c, d, x_0, \ldots, x_k)$.

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Suppose $w_0, \dots, w_k \in N$ are words such that $N = \langle\langle w_0, \dots, w_k \rangle\rangle^{\Gamma_k}[N, N]$.

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- at the boundary $\partial([0, 2\pi]^3 \times (\mathbb{D}^3)^k)$ it is even.

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Lemma

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ *be smooth. If* $f(a) = 0$ *and* $\det(Df_a) \neq 0$ *then* a *is an isolated zero and the degree of* f *around* a *is* $\deg(f, a) = \mathrm{sg}(\det(Df_a)) = \pm 1$.

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(iv) Let f, g be nowhere zero and assume $f(\mathbf{t}_0), g(\mathbf{t}_0) \in \{1, -1\}$. Then $[f, g](\mathbf{t}_0) = 1$ and

$$\frac{\partial [f, g]}{\partial t_i}(\mathbf{t}_0) = 0,$$

where $[f, g] = f \cdot g \cdot \frac{1}{f} \cdot \frac{1}{g}$ is the commutator.

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Let $I = [-1, 1]$ *and let* $\mathbb{D}^3 \subset \mathbb{R}^3$ *be the unit disk. Let*

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If $\mathbf{F}$ is nonzero on the boundary of $I^3 \times (\mathbb{D}^3)^k$, the degree of the restriction $\mathbf{F} \colon \partial(I^3 \times (\mathbb{D}^3)^k) \to (\mathbb{D}^3)^{k+1} - \{0\}$ is even.

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The extension

$$1 \to N \to \Gamma_k \xrightarrow{\overline{\phi}} A_5 \to 1$$

has a relation gap: the A_5-module $N/[N,N]$ is free of rank $k+1$, however the Γ_k-group N cannot be generated with $k+1$ elements.

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Let $P(G)$ be the following proposition: there is a nontrivial orthogonal representation of the fundamental group of every acyclic G-complex obtained from $\Gamma_{OS}(G)$ by attaching $k \geq 0$ free orbits of 1-cells and $(k+1)$ free orbits of 2-cells.

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- $\mathrm{PSL}_2(2^p)$ *for p prime;*
- $\mathrm{PSL}_2(3^p)$ *for an odd prime p;*
- $\mathrm{PSL}_2(q)$ *for a prime $q > 3$ such that $q \equiv \pm 3 \pmod 5$ and $q \equiv \pm 3 \pmod 8$;*
- $\mathrm{Sz}(2^p)$ *for p an odd prime.*

Then every action of a finite group on a finite and contractible 2-complex has a fixed point.

Beyond A_5

Let $P(G)$ be the following proposition: there is a nontrivial orthogonal representation of the fundamental group of every acyclic G-complex obtained from $\Gamma_{OS}(G)$ by attaching $k \geq 0$ free orbits of 1-cells and $(k+1)$ free orbits of 2-cells.

Using the results of Oliver–Segev and the Gerstenhaber–Rothaus theorem we can prove the following.

Theorem (S.C.)

Assume that $P(G)$ holds for each of the following groups G

- $\mathrm{PSL}_2(2^p)$ *for p prime;*
- $\mathrm{PSL}_2(3^p)$ *for an odd prime p;*
- $\mathrm{PSL}_2(q)$ *for a prime $q > 3$ such that $q \equiv \pm 3 \pmod 5$ and $q \equiv \pm 3 \pmod 8$;*
- $\mathrm{Sz}(2^p)$ *for p an odd prime.*

Then every action of a finite group on a finite and contractible 2-complex has a fixed point.

We know $P(A_5)$ holds. We are currently working on extending this result to other groups in this list.

Thank you!

Questions?

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