Relative I-test, asphericity and equations over groups

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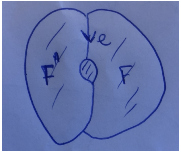
Recall that a combinatorial 2-complex K is diagrammatically reducible (DR) if every spherical diagram $f : C \to K$ contains a pair of opposite faces (= 2-cells with an edge e in common, which map to the same 2-cell of K wih opposite orientations folding over e).



A group presentation $P = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle$ is DR if the associated 2-complex K_P is.

(1) DR $\Rightarrow$ aspherical

(2) Applications to equations over groups

A system of equations over a group G with unknowns $x_1, x_2, \ldots, x_n$

$$S = \{w_j(x_1, x_2, \ldots, x_n)\}_j$$

words in $G * F(x_1, \ldots, x_n)$.

The letters of w_j which lie in G are the *coefficients* of w_j. The (non-necessarily reduced) word r_j in the x_i, x_i^{-1}, obtained by deleting the coefficients of w_j is the *shape* of w_j.

The system S has a solution in an overgroup of G if there exits a group H which contains G as a subgroup and elements $h_1, h_2, \ldots, h_n$ in H such that

$$w_j(h_1, h_2, \ldots, h_n) = 1 \in H$$

for every j.

Kervaire-Laudenbach Conjecture: for any group G, a unique equation w with a unique unknown x has a solution in an overgroup of G if w is non-singular ($=$ the total exponent of x is non-zero).

Kervaire-Laudenbach-Howie Conjecture: the same with a finite number n of unknowns and a non-singular system of m equations (non-singular means that the $m \times n$ matrix of total exponents has rank equal to m).

Let S be a system of equations $w_1, w_2, \ldots, w_m$ over a group G. Let P be the presentation $\langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ whose generators are the unknowns of S and its relators are the shapes of the equations w_j.

Theorem (Gersten)

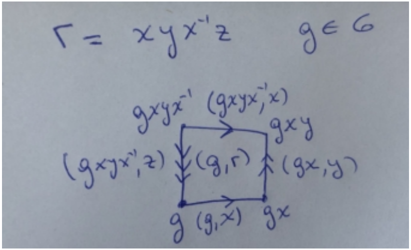
If P is DR, then for any group G, every system of equations S modelled by P has a solution in an overgroup of G.

The (absolute) I-test

Theorem (Corson-Trace) A combinatorial 2-complex K is DR if and only if every finite subcomplex of the universal cover $\tilde{K}$ collapses to a 1-complex.

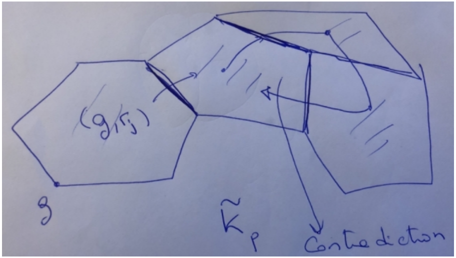
Strategy behind the I-test:

Given a presentation $P = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle$ of a group G. The idea is to find, if possible, a (partial) ordering of the 1-cells of $\tilde{K}_P$ (weights) and to match every relator r_j with some generator x_i (which is a letter in r_j) in such a way that the 2-cells in $\tilde{K}_P$ corresponding to r_j attain the maximum in a specific 1-cell of its boundary corresponding to a lift of x_i.



For example we match the relator r with generator x and the maximum is attained in the second occurrence of x (corresponding to the subword xyx^{-1}).

In that case we are done. Since if there exists a finite $L \subset \tilde{K}_P$ with no free faces, we arrive to a contradiction:



In the I-test the ordering and the matching are induced by a function (assignment of weights) $\varphi : G \to Q$ (Q poset) with the property that $\varphi(g) \leq \varphi(g')$ implies $\varphi(hg) \leq \varphi(hg')$. For example Q is a left orderable group and φ a group homomorphism. (The original I-test uses $Q = \mathbb{Z}$).

Given such a map, we construct a matrix of weights of the subwords corresponding to the occurrences of the generators in the relators.

$$P = \langle x, y, z, w | x^2y^2z^2, xyx^{-1}zyz^{-1}, w^2x^{-1}w^{-1}z \rangle$$

$$\begin{array}{c} \\ x \\ y \\ z \\ w \end{array} \begin{array}{c} \begin{array}{ccc} r_1 & r_2 & r_3 \end{array} \\ \begin{pmatrix} 0, -1 & 0, 0 & -3 \\ -2, -2 & -1, 1 & \emptyset \\ -2, -1 & 0, 0 & -1 \\ \emptyset & \emptyset & 0, -2, -1 \end{pmatrix} \end{array}$$

The matrix will be “good” if there is an ordering of the columns (relators) and for each column there is a row (generator) such that the corresponding entry has the maximum of the whole row and it is the unique maximum of the columns which are not already used.

$$
\begin{array}{c} \\ x \\ y \\ z \\ w \end{array}
\begin{array}{c}
\begin{array}{ccc} r_1 & r_2 & r_3 \end{array} \\
\begin{pmatrix}
0,-1 & 0,0 & -3 \\
-2,-2 & -1,1 & \emptyset \\
-2,-1 & 0,0 & -1 \\
\emptyset & \emptyset & 0,-2,-1
\end{pmatrix}
\end{array}
$$

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\end{array}
$$

A presentation $P = \langle x_1, x_2, \dots, x_n | r_1, r_2, \dots, r_m \rangle$ satisfies the *I-test* if there exists a map $G \to \mathbb{Z}$ such that the corresponding matrix is good.

Theorem (Barmak- M.) If P satisfies the I-test, it is DR.

Example: The following family of presentations satisfy the I-test

Let $n \in \mathbb{N}$ and let

$$P = \langle x_1, x_2, \dots, x_n, x | x_i x \omega_i = x \tau_i, 1 \leq i \leq n \rangle$$

where ω_i, τ_i are words in which x occurs only with positive exponent. Suppose further that for each i the total exponent $exp(x, \omega_i)$ coincides with $exp(x, \tau_i)$. Then P satisfies the I-test.

One can use the test to investigate the following asphericity conjecture of Ivanov:

Conjecture:Let $P = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ be an aspherical presentation and let $Q = \langle x_1, x_2, \ldots, x_n, x | r_1, r_2, \ldots, r_m, r \rangle$ be such that:

- The total exponent of x in r is non-zero,
- The group H presented by P embeds in the group G presented by Q, and
- G is torsion-free.

Then Q is aspherical.

Ivanov's conjecture is related to Kaplansky conjecture on zero divisors of a group ring. If Ivanov's Conjecture is false, there is a torsion free group G such that $\mathbb{Z}G$ has zero divisors.

In particular, it holds for groups with the unique product property (upp). Recall that a group G has the upp if for any two nonempty finite subsets $A, B \subset G$ there exists a $g \in G$ such that $gA \cap B$ has exactly one element.

Using a variation of our test, one can give an alternative proof that Ivanov's conjecture holds for groups with the upp.

We can also prove a kind of weak version of the conjecture (if we are allowed to perturb the relation r):

Theorem (Barmak- M.) Let $P = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ be an aspherical presentation of an indicable group H and let $Q = \langle x_1, x_2, \ldots, x_n, x | r_1, r_2, \ldots, r_m, r \rangle$ be such that the total exponent of x in r is non-zero. Then there exists a cyclic permutation r' of r, $1 \leq i \leq n$ and $M_0 \in \mathbb{N}$ such that, for every integer M with $|M| \geq M_0$, the perturbed presentation

$$Q' = \langle x_1, x_2, \ldots, x_n, x | r_1, r_2, \ldots, r_m, x_i^M r' \rangle$$

is aspherical if H naturally embeds in the group G' presented by Q'.

Applications to equations over groups

We focus on solutions of one equation w with many unknowns.

By a result attributed to Pride, for any coefficient group G, if the shape of w is cyclically reduced then the equation has a solution in an overgroup of G.

Recently Klyachko and Thom proved the following result concerning one equation with many variables over hyperlinear groups.

Theorem (Klyachko - Thom)

Let G be a hyperlinear group. An equation in two variables with coefficients in G can be solved over G if its content does not lie in $[F_2, [F_2, F_2]]$.

Using the I-test we can prove Kervaireness in many cases which are not covered by the results of Pride and Klyachko-Thom.

Proposition (Barmak-M.). Let $w_1, w_2, \dots, w_n$ be nontrivial words in two variables $\{x, y\}$, each of them positive or negative, and let $w = [w_n, [w_{n-1}, \dots, [w_2, [w_1, [x, y]]] \dots]]$. Then for any group H, any equation modeled by w has a solution in an overgroup of H.

The relative I-test

Relative presentations. Let H be a group. A relative presentation is $P = \langle H, x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle$. Each relator r_j is a word in $H * F$, where $F = F(x_1, x_2, \ldots, x_n)$. The group presented by P is $G = H * F / << r_1, \ldots, r_m >>$.

For studying solutions of equations over the group H, we want to give conditions on P such that the map $H \to G$ is injective.

The I-test can be adapted to the relative context: the idea is to find a morphism $\varphi : G \to Q$ (a poset, or a left orderable group, for example $\mathbb{Z}$) and construct a matrix as before where the columns correspond to the relators r_j, the rows to the generators x_i and the entries of the matrix are the values of φ in certain subwords of the relators determined by the x_i.

In the relative test. The notion of DR (and the result of Corson and Trace) is replaced by the following notion, which is a small generalization of a notion introduced by B. Bogley (in the context of LOGs).

Definition (B.Bogley). Let L be a subcomplex of a CW-complex K such that every cell of $K \setminus L$ is of dimension ≤ 2. We say that K *locally collapses to a* 1*-complex relative to* L if each finite subcomplex of K is either contained in $L \cup K^1$ or it has a free face of dimension 1 which is not in L.

The standard complex of P relative to L. Let $P = \langle H, x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ be a relative presentation of G. Let L be a connected CW-complex such that $\pi_1(L) = H$ and let $e^0 \in L$ a 0-cell.

Let K be the CW-complex obtained from L by attaching a 1-cell e_i^1 with endpoints in e^0 for each $1 \leq i \leq n$ and a 2-cell e_j^2 for each $1 \leq j \leq m$. The attaching map e_j^2 is a concatenation of loops in K^1 with endpoints in e^0, each of these loops corresponds to a letter of r_j. For the letters in H, choose a loop in L^1 representing that element in $\pi_1(L, e^0)$. For the letters $x_i^{\epsilon_i}$ the loop goes over the cell e_i^1.

We call K the *standard complex* of P relative to L.

The homotopy type of K is uniquely determined by P and L. Note that $\pi_1(K) = G$.

The following result was proved by Bogley for LOGs (labelled oriented graphs) and the universal abelian covering. The proof of this version is almost identical.

Theorem

Let $P = \langle H, x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ *be a relative presentation of a group* G*,* L *a connected CW-complex with* $\pi_1(L) = H$ *and let* K *be the standard complex of* P *relative to* L*. Let* $p : \widetilde{K} \to K$ *be the universal cover of* K*. If* $\widetilde{K}$ *locally collapses to a 1-complex relative to* $p^{-1}(L)$*, then the following hold:*
(i) The natural map $H \to G$ *is injective, and*
(ii) The map $\pi_q(L \cup K^1) \to \pi_q(K)$ *induced by the inclusion is surjective for* $q \geq 1$*.*

Proof.

By the long exact sequence of homotopy groups, an element in the kernel of $\pi_1(L) = H \to \pi_1(K) = G$ is represented, via $\partial : \pi_2(K, L) \to \pi_1(L)$ by a map $f : (D^2, S^1) \to (K, L)$. This map lifts to a map $\widetilde{f} : (D^2, S^1) \to (\widetilde{K}, p^{-1}(L))$ with image contained in a finite subcomplex of $\widetilde{K}$. By definition of local collapse and induction, $\widetilde{f}$ is homotopic relative to S^1 to a map $g : (D^2, S^1) \to (p^{-1}(L) \cup \widetilde{K}^1, p^{-1}(L))$, and then $f \simeq pg \text{ rel } S^1$. Then $\partial([f]) = \partial([pg])$ is in the kernel of $\pi_1(L) \to \pi_1(L \cup K^1)$, which is the inclusion in a free factor and then injective, so $\partial([f])$ is trivial. Thus, $H \to G$ is injective.

On the other hand, an element of $\pi_q(K)$ is represented by a map $f : (D^q, S^{q-1}) \to (K, e^0)$. By the same argument above, f is homotopic relative to S^{q-1} to a map $g : (D^q, S^{q-1}) \to (L \cup K^1, e^0)$, so $[f]$ is in the image of $\pi_q(L \cup K^1) \to \pi_q(K)$. □

Since asphericity is invariant by attachment of 1-cells, we have the following

Corollary. Under the hypotheses of Theorem above, if L is aspherical, then so is K.

Theorem (Barmak- M.) Let $P = \langle H, x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle$ be a relative presentation of a group G, L a connected CW-complex with $\pi_1(L) = H$ and let K be the standard complex of P relative to L. Let $p : \widetilde{K} \to K$ be the universal cover of K. If P satisfies the I-test, then $\widetilde{K}$ locally collapses to a 1-complex relative to $p^{-1}(L)$.

Corollary. If $P = \langle H, x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle$ satisfies the I-test then it is aspherical (in the sense that $\pi_2(K, L) = 0$ for $L = K(H, 1)$ and K the standard complex relative to L). In particular, the map $H \to G$ is injective.

THANKS!!